DIFFERENTIAL CALCULUS

BY

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PREFACE

In this text on differential calculus I have continued the plan adopted for my Analytic Geometry, wherein a few central methods are expounded and applied to a large variety of examples to the end that the student may learn principles and gain power. In this way the differential calculus makes only a brief text suitable for a term's work and leaves for the integral calculus, which in many respects is far more important, a greater proportion of time than is ordinarily devoted to it.

As material for review and to provide problems for which answers are not given, a supplementary list, containing about half as many exercises as occur in the text, is placed at the end of the book.

I wish to acknowledge my indebtedness to Professor H. W. Tyler and Professor E. B. Wilson for advice and criticism and to Dr. Joseph Lipka for valuable assistance in preparing the manuscript and revising the proof.

H. B. PHILLIPS.

Boston, Mass., *August*, 1916.

CONTENTS

DIFFERENTIAL CALCULUS

CHAPTER I

INTRODUCTION

1. Definition of Function. — *A quantity y is called a function of a quantity x if values of y are determined by values of x.*

Thus, if $y = 1 - x^2$, y is a function of x; for a value of x determines a value of y. Similarly, the area of a circle is a function of its radius; for, the radius being given, the area is determined.

It is not necessary that only one value of the function correspond to a value of the variable. Several values may be determined. Thus, if x and y satisfy the equation

$$x^2 - 2\,xy + y^2 = x,$$

then y is a function of x. To each value of x correspond two values of y found by solving the equation for y.

A quantity u is called a function of several variables if u is determined when values are assigned to those variables.

Thus, if $z = x^2 + y^2$, then z is a function of x and y; for, values being given to x and y, a value of z is determined. Similarly, the volume of a cone is a function of its altitude and radius of base; for the radius and altitude being assigned, the volume is determined.

2. Kinds of Functions. — An expression containing variables is called an *explicit* function of those variables. Thus $\sqrt{x + y}$ is an explicit function of x and y. Similarly, if

$$y = \sqrt{x + 1},$$

y is an explicit function of x.

A quantity determined by an equation not solved for that quantity is called an *implicit* function. Thus, if

$$x^2 - 2\,xy + y^2 = x,$$

y is an implicit function of x. Also x is an implicit function of y.

Explicit and implicit do not denote properties of the function but of the way it is expressed. An implicit function is rendered explicit by solving. For example, the above equation is equivalent to

$$y = x \pm \sqrt{x},$$

in which y appears as an explicit function of x.

A *rational* function is one representable by an algebraic expression containing no fractional powers of variable quantities. For example,

$$\frac{x\sqrt{5} + 3}{x^2 + 2\,x}$$

is a rational function of x.

An *irrational* function is one represented by an algebraic expression which cannot be reduced to rational form. Thus $\sqrt{x + y}$ is an irrational function of x and y.

A function is called *algebraic* if it can be represented by an algebraic expression or is the solution of an algebraic equation. All the functions previously mentioned are algebraic.

Functions that are not algebraic are called *transcendental.* For example, $\sin x$ and $\log x$ are transcendental functions of x.

3. Independent and Dependent Variables. — In most problems there occur a number of variable quantities connected by equations. Arbitrary values can be assigned to some of these quantities and the others are then determined. Those taking arbitrary values are called *independent* variables; those determined are called *dependent* variables. Which variables are taken as independent and which as dependent is usually a matter of convenience. The number of independent variables is, however, determined by the equations.

For example, in plotting the curve

$$y = x^3 + x,$$

values are assigned to x and values of y are calculated. The independent variable is x and the dependent variable y. We might assign values to y and calculate values of x but that would be much more difficult.

4. Notation. — A particular function of x is often represented by the notation $f(x)$, which should be read, function of x, or f of x, not f times x. For example,

$$f(x) = \sqrt{x^2 + 1}$$

means that $f(x)$ is a symbol for $\sqrt{x^2+1}$. Similarly,

$$y = f(x)$$

means that y is some definite (though perhaps unknown) function of x.

If it is necessary to consider several functions in the same discussion, they are distinguished by subscripts or accents or by the use of different letters. Thus, $f_1(x)$, $f_2(x)$, $f'(x)$, $f''(x)$, $g(x)$ (read f-one of x, f-two of x, f-prime of x, f-second of x, g of x) represent (presumably) different functions of x.

Functions of several variables are expressed by writing commas between the variables. For example,

$$v = f(r, h)$$

expresses that v is a function of r and h and

$$v = f(a, b, c)$$

expresses that v is a function of a, b, c.

The f in the symbol of a function should be considered as representing an operation to be performed on the variable or variables. Thus, if

$$f(x) = \sqrt{x^2 + 1},$$

f represents the operation of squaring the variable, adding 1, and extracting the square root of the result. If x is replaced by any other quantity, the same operation is to be performed on that quantity. For example,

$$f(2) = \sqrt{2^2 + 1} = \sqrt{5}.$$
$$f(y+1) = \sqrt{(y+1)^2 + 1} = \sqrt{y^2 + 2y + 2}.$$

Similarly, if

$$f(x, y) = x^2 + xy - y^2,$$

then $$f(1, 2) = 1^2 + 1 \cdot 2 - 2^2 = -1.$$

If $$f(x, y, z) = x^2 + y^2 + z^2,$$

then $$f(2, -3, 1) = 2^2 + (-3)^2 + 1 = 14.$$

EXERCISES

1. Given $x^{\frac{3}{2}} + y^{\frac{3}{2}} = a^{\frac{3}{2}}$, express y as an explicit function of x.

2. Given $\log_{10}(x) = \sin y$, express x as an explicit function of y. Also express y as an explicit function of x.

3. If $f(x) = x^2 - 3x + 2$, show that $f(1) = f(2) = 0$.

4. If $F(x) = x^4 + 2x^2 + 3$, show that $F(-a) = F(a)$.

5. If $F(x) = x + \dfrac{1}{x}$, find $F(x+1)$. Also find $F(x) + 1$.

6. If $\phi(x) = \sqrt{x^2 - 1}$, find $\phi(2x)$. Also find $2\phi(x)$.

7. If $\psi(x) = \dfrac{x+2}{2x-3}$, find $\psi\left(\dfrac{1}{x}\right)$. Also find $\dfrac{1}{\psi(x)}$.

8. If $f_1(x) = 2^x$, $f_2(x) = x^2$, find $f_1[f_2(y)]$. Also find $f_2[f_1(y)]$.

9. If $f(x, y) = x - \dfrac{1}{y}$, show that $f(2, 1) = 2f(1, 2) = 1$.

10. Given $f(x, y) = x^2 + xy$, find $f(y, x)$.

11. On how many independent variables does the volume of a right circular cylinder depend?

12. Three numbers x, y, z satisfy two equations

$$x^2 + y^2 + z^2 = 5,$$
$$x + y + z = 1.$$

How many of these numbers can be taken as independent variables?

5. Limit. — If in any process a variable quantity approaches a constant one in such a way that the difference of the two becomes and remains as small as you please, the constant is said to be the *limit* of the variable.

The use of limits is well illustrated by the incommensurable cases of geometry and the determination of the area of a circle or the volume of a cone or sphere.

6. Limit of a Function. — As a variable approaches a limit a function of that variable may approach a limit. Thus, as x approaches 1, $x^2 + 1$ approaches 2.

We shall express that a variable x approaches a limit a by the notation

$$x \doteq a.$$

The symbol $\doteq$ thus means "approaches as a limit."

Let $f(x)$ approach the limit A as x approaches a; this is expressed by

$$\lim_{x \doteq a} f(x) = A,$$

which should be read, "the limit of $f(x)$, as x approaches a, is A."

Example 1. Find the value of

$$\lim_{x \doteq 1}\left(x + \frac{1}{x}\right).$$

As x approaches 1, the quantity $x + \frac{1}{x}$ approaches $1 + \frac{1}{1}$ or 2. Hence

$$\lim_{x \doteq 1}\left(x + \frac{1}{x}\right) = 2.$$

Ex. 2. Find the value of

$$\lim_{\theta \doteq 0} \frac{\sin\theta}{1 + \cos\theta}.$$

As θ approaches zero, the function given approaches

$$\frac{0}{1+1} = 0.$$

Hence

$$\lim_{\theta \doteq 0} \frac{\sin\theta}{1 + \cos\theta} = 0.$$

7. Properties of Limits. — In finding the limits of functions frequent use is made of certain simple properties that follow almost immediately from the definition.

1. *The limit of the sum of a finite number of functions is equal to the sum of their limits.*

Suppose, for example, X, Y, Z are three functions approaching the limits A, B, C respectively. Then $X+Y+Z$ is approaching $A+B+C$. Consequently,

$$\lim (X+Y+Z) = A+B+C = \lim X + \lim Y + \lim Z.$$

2. *The limit of the product of a finite number of functions is equal to the product of their limits.*

If, for example, X, Y, Z approach A, B, C respectively, then XYZ approaches ABC, that is,

$$\lim XYZ = ABC = \lim X \lim Y \lim Z.$$

3. *If the limit of the denominator is not zero, the limit of the ratio of two functions is equal to the ratio of their limits.*

Let X, Y approach the limits A, B and suppose B is not zero. Then $\frac{X}{Y}$ approaches $\frac{A}{B}$, that is,

$$\lim \frac{X}{Y} = \frac{A}{B} = \frac{\lim X}{\lim Y}.$$

If B is zero and A is not zero, $\frac{A}{B}$ will be infinite. Then $\frac{X}{Y}$ cannot approach $\frac{A}{B}$ as a limit; for, however large $\frac{X}{Y}$ may become, the difference of $\frac{X}{Y}$ and infinity will not become small.

8. The Form $\frac{0}{0}$. — When x is replaced by a particular value, a function sometimes takes the form $\frac{0}{0}$. Although this symbol does not represent a definite value, the function may have a definite limit. This is usually made evident by writing the function in a different form.

Example 1. Find the value of

$$\lim_{x \doteq 1} \frac{x^2 - 1}{x - 1}$$

When x is replaced by 1, the function takes the form

$$\frac{1-1}{1-1} = \frac{0}{0}.$$

Since, however,

$$\frac{x^2-1}{x-1} = x + 1,$$

the function approaches $1 + 1$ or 2. Therefore

$$\lim_{x \doteq 1} \frac{x^2-1}{x-1} = 2.$$

Ex. 2. Find the value of

$$\lim_{x \doteq 0} \frac{(\sqrt{1+x} - 1)}{x}.$$

When $x = 0$ the given function becomes

$$\frac{1-1}{0} = \frac{0}{0}.$$

Multiplying numerator and denominator by $\sqrt{1+x} + 1$,

$$\frac{\sqrt{1+x}-1}{x} = \frac{x}{x\,(\sqrt{1+x}+1)} = \frac{1}{\sqrt{1+x}+1}.$$

As x approaches 0, the last expression approaches $\frac{1}{2}$. Hence

$$\lim_{x \doteq 0} \frac{(\sqrt{1+x} - 1)}{x} = \frac{1}{2}.$$

9. Infinitesimal. — A variable approaching zero as a limit is called an *infinitesimal.*

Let α and β be two infinitesimals. If

$$\lim \frac{\alpha}{\beta}$$

is finite and not zero, α and β are said to be infinitesimals of the same order. If the limit is zero, α is of higher order than β. If the ratio $\frac{\alpha}{\beta}$ approaches infinity, β is of higher order than α. Roughly speaking, the higher the order, the smaller the infinitesimal.

For example, let x approach zero. The quantities

$$x,\ x^2,\ x^3,\ x^4, \text{ etc.}$$

are infinitesimals arranged in ascending order. Thus x^4 is of higher order than x^2; for

$$\lim_{x \doteq 0} \frac{x^4}{x^2} = \lim_{x \doteq 0} x^2 = 0.$$

Similarly, x^3 is of lower order than x^4, since

$$\frac{x^3}{x^4} = \frac{1}{x}$$

approaches infinity when x approaches zero.

As x approaches $\frac{\pi}{2}$, $\cos x$ and $\cot x$ are infinitesimals of the same order; for

$$\lim_{x \doteq \frac{\pi}{2}} \frac{\cos x}{\cot x} = \lim_{x \doteq 0} \sin x = 1,$$

which is finite and not zero.

EXERCISES

Find the values of the following limits:

1. $\lim\limits_{x \doteq 0} \dfrac{x^2 - 2x + 3}{x - 5}$.

2. $\lim\limits_{\theta \doteq \frac{\pi}{4}} \dfrac{\sin \theta + \cos \theta}{\sin 2\theta + \cos 2\theta}$.

3. $\lim\limits_{x \doteq 1} \dfrac{x^2 - 3x + 2}{x - 1}$.

4. $\lim\limits_{x \doteq 0} \dfrac{\sqrt{1 - x^2} - \sqrt{1 + x^2}}{x^2}$.

5. $\lim\limits_{\theta \doteq 0} \dfrac{\sin \theta}{\tan \theta}$.

6. $\lim\limits_{\theta \doteq 0} \dfrac{\sin \theta}{\sin 2\theta}$.

7. By the use of a table of natural sines find the value of

$$\lim_{x \doteq 0} \frac{\sin x}{x}.$$

8. Define as a limit the area within a closed curve.

9. Define as a limit the volume within a closed surface.

10. Define $\sqrt{2}$.

11. On the segment PQ (Fig. 9a) construct a series of equilateral triangles reaching from P to Q. As the number of triangles is increased,

their bases approaching zero, the polygonal line *PABC*, etc., approaches *PQ*. Does its length approach that of *PQ*?

FIG. 9a.

12. Inscribe a series of cylinders in a cone as shown in Fig. 9b. As the number of cylinders increases indefinitely, their altitudes approaching zero, does the sum of the volumes of the cylinders approach that of the cone? Does the sum of the lateral areas of the cylinders approach the lateral area of the cone?

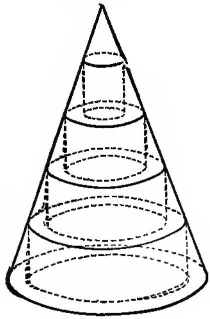
FIG. 9b.

13. Show that when x approaches zero, $\tan \frac{1}{x}$ does not approach a limit.

14. As x approaches 1, which of the infinitesimals $1 - x$ and $\sqrt{1 - x}$ is of higher order?

15. As the radius of a sphere approaches zero, show that its volume is an infinitesimal of higher order than the area of its surface and of the same order as the volume of the circumscribing cylinder.

CHAPTER II

DERIVATIVE AND DIFFERENTIAL

10. Increment. — When a variable changes value, the algebraic increase (new value minus old) is called its *increment* and is represented by the symbol Δ written before the variable.

Thus, if x changes from 2 to 4, its increment is

$$\Delta x = 4 - 2 = 2.$$

If x changes from 2 to -1,

$$\Delta x = -1 - 2 = -3.$$

When the increment is positive there is an increase in value, when negative a decrease.

Let y be a function of x. When x receives an increment Δx, an increment Δy will be determined. The increments of x and y thus correspond. To illustrate this graphically let x and y be the rectangular coördinates of a point P. An equation

$$y = f(x)$$

represents a curve. When x changes, the point P changes to some other position Q on the curve. The increments of x and y are

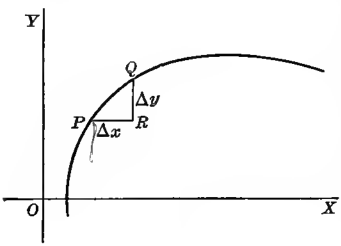

FIG. 10.

$$\Delta x = PR, \qquad \Delta y = RQ. \tag{10}$$

11. Continuous Function. — A function is called *continuous* if the increment of the function approaches zero as the increment of the variable approaches zero.

In Fig. 10, y is a continuous function of x; for, as Δx approaches zero, Q approaches P and so Δy approaches zero.

In Figs. 11a and 11b are shown two ways that a function can be *discontinuous*. In Fig. 11a the curve has a break at

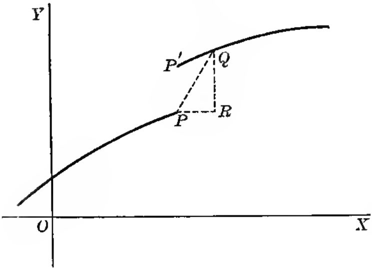

FIG. 11a.

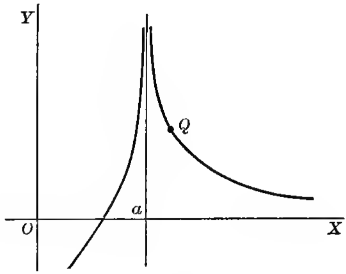

FIG. 11b.

P. As Q approaches P', $\Delta x = PR$ approaches zero, but $\Delta y = RQ$ does not. In Fig. 11b the ordinate at $x = a$ is infinite. The increment Δy occurring in the change from $x = a$ to any neighboring value is infinite.

12. Slope of a Curve. — As Q moves along a continuous curve toward P, the line PQ turns about P and usually approaches a limiting position PT. This line PT is called the *tangent* to the curve at P.

The slope of PQ is

$$\frac{RQ}{PR} = \frac{\Delta y}{\Delta x}.$$

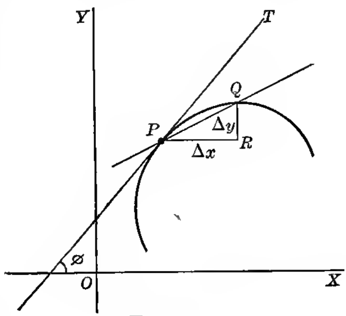

FIG. 12a.

As Q approaches P, Δx approaches zero and the slope of PQ approaches that of PT. Therefore

$$\text{Slope of the tangent} = \tan\phi = \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x}. \qquad (12)$$

The slope of the tangent at P is called the *slope of the curve* at P.

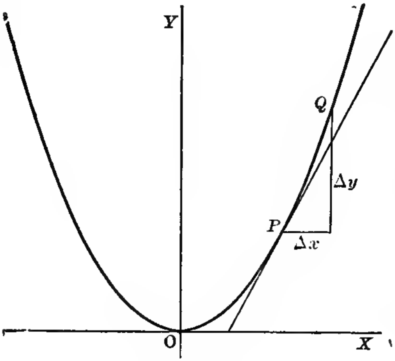

FIG. 12b.

Example. Find the slope of the parabola $y = x^2$ at the point (1, 1).

Let the coördinates of P be x, y. Those of Q are $x + \Delta x$, $y + \Delta y$. Since P and Q are both on the curve,

$$y = x^2$$

and

$$y + \Delta y = (x + \Delta x)^2 = x^2 + 2\,x\,\Delta x + (\Delta x)^2.$$

Subtracting these equations, we get

$$\Delta y = 2\,x\,\Delta x + (\Delta x)^2.$$

Dividing by Δx,

$$\frac{\Delta y}{\Delta x} = 2\,x + \Delta x.$$

As Δx approaches zero, this approaches

$$\text{Slope at } P = 2\,x.$$

This is the slope at the point with abscissa x. The slope at (1, 1) is then $2 \cdot 1 = 2$.

13. Derivative. — Let y be a function of x. If $\dfrac{\Delta y}{\Delta x}$ approaches a limit as Δx approaches zero, that limit is called the *derivative of y with respect to x.* It is represented by the notation $D_x y$, that is,

$$D_x y = \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x}. \tag{13a}$$

If a function is represented by $f(x)$, its derivative with respect to x is often represented by $f'(x)$. Thus

$$f'(x) = \lim_{\Delta x \doteq 0} \frac{\Delta f(x)}{\Delta x} = D_x f(x). \tag{13b}$$

In Art. 12 we found that this limit represents the slope of the curve $y = f(x)$. The derivative is, in fact, a function of x whose value is the slope of the curve at the point with abscissa x.

The derivative, being the limit of $\frac{\Delta y}{\Delta x}$, is approximately equal to a small change in y divided by the corresponding small change in x. It is then large or small according as the small increment of y is large or small in comparison with that of x.

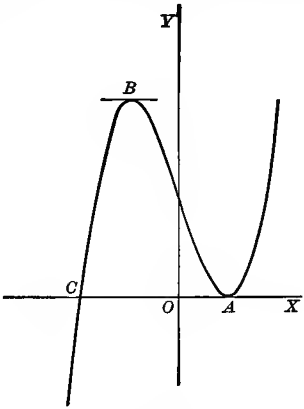

FIG. 13.

If small increments of x and y have the same sign $\frac{\Delta y}{\Delta x}$ and its limit $D_x y$ are positive. If they have opposite signs $D_x y$ is negative. Therefore $D_x y$ is positive when x and y increase and decrease together and negative when one increases as the other decreases.

Example. $y = x^3 - 3x + 2$.

Let x receive an increment Δx. The new value of x is $x + \Delta x$. The new value of y is $y + \Delta y$. Since these satisfy the equation,

$$y + \Delta y = (x + \Delta x)^3 - 3(x + \Delta x) + 2.$$

Subtracting the equation

$$y = x^3 - 3x + 2,$$

we get

$$\Delta y = 3x^2 \Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 3\Delta x.$$

Dividing by Δx,

$$\frac{\Delta y}{\Delta x} = 3x^2 + 3x\Delta x + (\Delta x)^2 - 3.$$

As Δx approaches zero this approaches the limit

$$D_x y = 3x^2 - 3.$$

The graph is shown in Fig. 13. At A (where $x = 1$) $y = 0$ and $D_xy = 3 \cdot 1 - 3 = 0$. The curve is thus tangent to the x-axis at A. The slope is also zero at B (where $x = -1$). This is the highest point on the arc AC. On the right of A and on the left of B, the slope D_xy is positive and x and y increase and decrease together. Between A and B the slope is negative and y decreases as x increases.

EXERCISES

1. Given $y = \sqrt{x}$, find the increment of y when x changes from $x = 2$ to $x = 1.9$. Show that the increments approximately satisfy the equation

$$\frac{\Delta y}{\Delta x} = \frac{1}{2\sqrt{x}}.$$

2. Given $y = \log_{10} x$, find the increments of y when x changes from 50 to 51 and from 100 to 101. Show that the second increment is approximately half the first.

3. The equation of a certain line is $y = 2x + 3$. Find its slope by calculating the limit of $\frac{\Delta y}{\Delta x}$.

4. Construct the parabola $y = x^2 - 2x$. Show that its slope at the point with abscissa x is $2(x - 1)$. Find its slope at (4, 8). At what point is the slope equal to 2?

5. Construct the curve represented by the equation $y = x^4 - 2x^2$. Show that its slope at the point with abscissa x is $4x(x^2 - 1)$. At what points are the tangents parallel to the x-axis? Indicate where the slope is positive and where negative.

In each of the following exercises show that the derivative has the value given. Also find the slope of the corresponding curve at $x = -1$.

6. $y = (x + 1)(x + 2)$, $D_xy = 2x + 3$.

7. $y = x^4$, $D_xy = 4x^3$.

8. $y = x^3 - x^2$, $D_xy = 3x^2 - 2x$.

9. $y = \frac{1}{x}$, $D_xy = -\frac{1}{x^2}$.

10. If x is an acute angle, is $D_x \cos x$ positive or negative?

11. For what angles is $D_x \sin x$ positive and for what angles negative?

14. Approximate Value of the Increment of a Function.— Let y be a function of x and represent by ϵ a quantity such that

$$\frac{\Delta y}{\Delta x} = D_xy + \epsilon.$$

As Δx approaches zero, $\dfrac{\Delta y}{\Delta x}$ approaches $D_x y$ and so ϵ approaches zero.

The increment of y is

$$\Delta y = D_x y \, \Delta x + \epsilon \Delta x.$$

The part

$$D_x y \, \Delta x \tag{14}$$

is called the *principal part* of Δy. It differs from Δy by an amount $\epsilon \Delta x$. As Δx approaches zero, ϵ approaches zero, and so $\epsilon \Delta x$ becomes an indefinitely small fraction of Δx. It is an infinitesimal of higher order than Δx. If then the principal part is used as an approximation for Δy, the error will be only a small fraction of Δx when Δx is sufficiently small.

Example. When x changes from 2 to 2.1 find an approximate value for the change in $y = \dfrac{1}{x}\cdot$

In exercise 9, page 14, the derivative of $\dfrac{1}{x}$ was found to be $-\dfrac{1}{x^2}\cdot$ Hence the principal part of Δy is

$$-\frac{1}{x^2}\Delta x = -\frac{1}{4}(.1) = -0.0250.$$

The exact increment is

$$\Delta y = \frac{1}{(2.1)^2} - \frac{1}{2^2} = -0.0232.$$

The principal part represents Δy with an error less than 0.002 which is 2% of Δx.

15. Differentials. — Let x be the independent variable and let y be a function of x. The principal part of Δy is called the *differential* of y and is denoted by dy; that is,

$$dy = D_x y \, \Delta x. \tag{15a}$$

This equation defines the differential of any function y of x. In particular, if $y = x$, $D_x y = 1$, and so

$$dx = \Delta x, \tag{15b}$$

that is, *the differential of the independent variable is equal to* *its increment and the differential of any function y is equal to the product of its derivative and the increment of the independent variable.*

Combining 15a and 15b, we get

$$dy = D_x y \, dx, \tag{15c}$$

whence

$$\frac{dy}{dx} = D_x y, \tag{15d}$$

that is, *the quotient* $\frac{dy}{dx}$ *is equal to the derivative of* y *with respect to* x.

Since $D_x y$ is the slope of the curve $y = f(x)$, equations 15b and 15c express that dy and dx are the sides of the right triangle PRT (Fig. 15) with hypotenuse PT extending along the tangent at P. On this diagram, Δx and Δy are the increments

$$\Delta x = PR, \qquad \Delta y = RQ,$$

occurring in the change from P to Q. The differentials are

$$dx = PR, \qquad dy = RT.$$

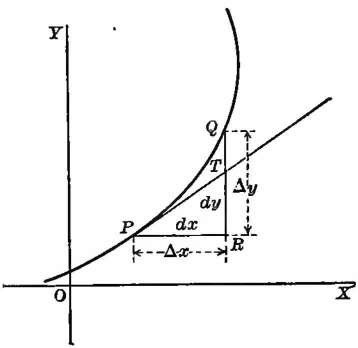

FIG. 15.

A point describing the curve is moving when it passes through P in the direction of the tangent PT. The differential dy is then the amount y *would increase* when x changes to $x + \Delta x$ if the direction of motion did not change. In general the direction of motion does change and so the actual increase $\Delta y = RQ$ is different from dy. If the increments are small the change in direction will be small and so Δy and dy will be approximately equal.

Equation 15c was obtained under the assumption that x was the independent variable. It is still valid if x and y are continuous functions of an independent variable t. For then

$$dx = D_t x \, \Delta t, \qquad dy = D_t y \, \Delta t.$$

The identity

$$\frac{\Delta y}{\Delta t} = \frac{\Delta y}{\Delta x} \cdot \frac{\Delta x}{\Delta t}$$

gives in the limit

$$D_t y = D_x y \cdot D_t x.$$

Hence

$$D_t y \, \Delta t = D_x y \cdot D_t x \, \Delta t,$$

that is,

$$dy = D_x y \, dx.$$

Example 1. Given $y = \dfrac{x+1}{x}$, find dy.

In this case

$$\Delta y = \frac{x + \Delta x + 1}{x + \Delta x} - \frac{x+1}{x} = -\frac{\Delta x}{x\,(x + \Delta x)}.$$

Consequently,

$$\frac{\Delta y}{\Delta x} = -\frac{1}{x\,(x + \Delta x)}.$$

As Δx approaches zero, this approaches

$$\frac{dy}{dx} = -\frac{1}{x^2}.$$

Therefore

$$dy = -\frac{dx}{x^2}.$$

Ex. 2. Given $x = t^2$, $y = t^3$, find $\dfrac{dy}{dx}$.

The differentials of x and y are found to be

$$dx = 2\,t\,dt, \qquad dy = 3\,t^2\,dt.$$

Division then gives,

$$\frac{dy}{dx} = \frac{3}{2}t.$$

Ex. 3. An error of 1% is made in measuring the side of a square. Find approximately the error in the calculated area.

Let x be the correct measure of the side and $x + \Delta x$ the value found by measurement. Then $dx = \Delta x = \pm 0.01\,x$.

The error in the area is approximately

$$dA = d\,(x^2) = 2\,x\,dx = \pm 0.02\,x^2 = \pm 0.02\,A,$$

which is 2% of the area.

EXERCISES

1. Let n be a positive integer and $y = x^n$. Expand

$$\Delta y = (x + \Delta x)^n - x^n$$

by using the binomial theorem. Show that

$$\frac{dy}{dx} = nx^{n-1}.$$

What is the principal part of Δy?

2. Using the results of Ex. 1, find an approximate value for the increment of x^6 when x changes from 1.1 to 1.2. Express the error as a percentage of Δx.

3. If A is the area of a circle of radius r, show that $\frac{dA}{dr}$ is equal to the circumference.

4. If the radius of a circle is measured and its area calculated by using the result, show that an error of 1% in the measurement of the radius will lead to an error of about 2% in the area.

5. If v is the volume of a sphere with radius r, show that $\frac{dv}{dr}$ is equal to the area of its surface.

6. Let v be the volume of a cylinder with radius r and altitude h. Show that if r is constant $\frac{dv}{dh}$ is equal to the area of the base of cylinder and if h is constant $\frac{dv}{dr}$ is equal to the lateral area.

7. If $y = f\,(x)$ and for all variations in x, $dx = \Delta x$, $dy = \Delta y$, show that the graph of $y = f\,(x)$ is a straight line.

8. If y is the independent variable and $x = f\,(y)$, make a diagram showing dx, dy, Δx, and Δy.

9. If the y-axis is vertical, the x-axis horizontal, a body thrown horizontally from the origin with a velocity of 50 ft. per second will in t seconds reach the point

$$x = 50\,t, \qquad y = -16\,t^2.$$

Find the slope of its path at that point.

10. A line turning about a fixed point P intersects the x-axis at A and the y-axis at B. If K_1 and K_2 are the areas of the triangles OPA and OPB, show that

$$\frac{dK_1}{dK_2} = \frac{PA^2}{PB^2}.$$

CHAPTER III

DIFFERENTIATION OF ALGEBRAIC FUNCTIONS

16. The process of finding derivatives and differentials is called *differentiation.* Instead of applying the direct method of the last chapter, differentiation is usually performed by means of certain formulas derived by that method. In this work we use the letter d for the operation of taking the differential and the symbol $\frac{d}{dx}$ for the operation of taking the derivative with respect to x. Thus

$$d(u+v) = \text{differential of } (u+v),$$

$$\frac{d}{dx}(u+v) = \text{derivative of } (u+v) \text{ with respect to } x.$$

To obtain the derivative with respect to x we proceed as in finding the differential except that d is everywhere replaced by $\frac{d}{dx}$.

17. Formulas. — Let u, v, w be continuous functions of a single variable x, and c, n constants.*

I. $$dc = 0.$$

II. $$d(u+v) = du + dv.$$

III. $$d(cu) = c\,du.$$

IV. $$d(uv) = u\,dv + v\,du.$$

V. $$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}.$$

VI. $$d(u^n) = nu^{n-1}\,du.$$

* It is assumed that the functions u, v, w have derivatives. There exist continuous functions,

$$u = f(x),$$

18. Proof of I. — *The differential of a constant is zero.*

When a variable x takes an increment Δx, a constant does not vary. Consequently, $\Delta c = 0$, $\dfrac{\Delta c}{\Delta x} = 0$, and in the limit $\dfrac{dc}{dx} = 0$. Clearing of fractions,

$$dc = dx \cdot 0 = 0.$$

19. Proof of II. — *The differential of the sum of a finite number of functions is equal to the sum of their differentials.*

Let

$$y = u + v.$$

When x takes an increment Δx, u will change to $u + \Delta u$, v to $v + \Delta v$, and y to $y + \Delta y$. Consequently

$$y + \Delta y = u + \Delta u + v + \Delta v.$$

Subtraction of the two equations gives

$$\Delta y = \Delta u + \Delta v,$$

whence

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}.$$

As Δx approaches zero, $\dfrac{\Delta y}{\Delta x}, \dfrac{\Delta u}{\Delta x}, \dfrac{\Delta v}{\Delta x}$ approach $\dfrac{dy}{dx}, \dfrac{du}{dx}, \dfrac{dv}{dx}$ respectively. Therefore

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx},$$

and so

$$dy = du + dv.$$

By the same method we can prove

$$d\,(u \pm v \pm w \pm \cdots) = du \pm dv \pm dw \pm \cdots.$$

such that

$$\frac{\Delta u}{\Delta x}$$

does not approach a limit as Δx approaches zero. Such a function has no derivative $D_x u$ and therefore no differential

$$du = D_x u\, dx.$$

20. Proof of III. — *The differential of a constant times a function is equal to the constant times the differential of the function.*

Let $$y = cu.$$

Then $$y + \Delta y = c\,(u + \Delta u)$$

and so $$\Delta y = c\,\Delta u,$$

$$\frac{\Delta y}{\Delta x} = c\,\frac{\Delta u}{\Delta x}.$$

As Δx approaches zero, $\frac{\Delta y}{\Delta x}$, and $c\,\frac{\Delta u}{\Delta x}$ approach $\frac{dy}{dx}$ and $c\,\frac{du}{dx}$. Therefore

$$\frac{dy}{dx} = c\,\frac{du}{dx},$$

whence

$$dy = c\,du.$$

Fractions with a constant denominator should be differentiated by this formula. Thus

$$d\left(\frac{u}{c}\right) = d\left(\frac{1}{c}u\right) = \frac{1}{c}\,du.$$

21. Proof of IV. — *The differential of the product of two functions is equal to the first times the differential of the second plus the second times the differential of the first.*

Let $$y = uv.$$

Then $$y + \Delta y = (u + \Delta u)\,(v + \Delta v)$$
$$= uv + v\,\Delta u + (u + \Delta u)\,\Delta v.$$

Subtraction gives

$$\Delta y = v\,\Delta u + (u + \Delta u)\,\Delta v,$$

whence

$$\frac{\Delta y}{\Delta x} = v\,\frac{\Delta u}{\Delta x} + (u + \Delta u)\,\frac{\Delta v}{\Delta x}.$$

Since u is a continuous function, Δu approaches zero as Δx approaches zero. Therefore, in the limit,

$$\frac{dy}{dx} = v\,\frac{du}{dx} + u\,\frac{dv}{dx},$$

and so

$$dy = v\,du + u\,dv.$$

In the same way we can show that

$$d\,(uvw) = uv\,dw + uw\,dv + vw\,du.$$

22. Proof of V. — *The differential of a fraction is equal to the denominator times the differential of the numerator minus the numerator times the differential of the denominator, all divided by the square of the denominator.*

Let

$$y = \frac{u}{v}.$$

Then

$$y + \Delta y = \frac{u + \Delta u}{v + \Delta v}$$

and

$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{v\,\Delta u - u\,\Delta v}{v\,(v + \Delta v)}.$$

Dividing by Δx,

$$\frac{\Delta y}{\Delta x} = \frac{v\dfrac{\Delta u}{\Delta x} - u\dfrac{\Delta v}{\Delta x}}{v\,(v + \Delta v)}.$$

Since v is a continuous function of x, Δv approaches zero as Δx approaches zero. Therefore

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2},$$

whence

$$dy = \frac{v\,du - u\,dv}{v^2}.$$

23. Proof of VI. — *The differential of a variable raised to a constant power is equal to the product of the exponent, the variable raised to a power one less, and the differential of the variable.*

We consider three cases depending on whether the exponent is a positive whole number, a positive fraction, or a negative number. For the case of irrational exponent, see Ex. 25, page 61.

(1) Let n be a positive integer and $y = u^n$. Then

$$y + \Delta y = (u + \Delta u)^n = u^n + nu^{n-1}\Delta u + \frac{n(n-1)}{2}u^{n-2}(\Delta u)^2 + \cdots$$

and

$$\Delta y = nu^{n-1}\,\Delta u + \frac{n\,(n-1)}{2}\,u^{n-2}\,(\Delta u)^2 + \cdots.$$

Dividing by Δu,

$$\frac{\Delta y}{\Delta u} = nu^{n-1} + \frac{n\,(n-1)}{2}\,u^{n-2}\,(\Delta u) + \cdots.$$

As Δu approaches zero, this approaches

$$\frac{dy}{du} = nu^{n-1}.$$

Consequently,

$$dy = nu^{n-1}\,du.$$

(2) Let n be a positive fraction $\frac{p}{q}$ and $y = u^n = u^{\frac{p}{q}}$. Then

$$y^q = u^p.$$

Since p and q are both positive integers, we can differentiate both sides of this equation by the formula just proved. Therefore

$$qy^{q-1}\,dy = pu^{p-1}\,du.$$

Solving for dy and substituting $u^{\frac{p}{q}}$ for y, we get

$$dy = \frac{pu^{p-1}}{qu^{p-\frac{p}{q}}}\,du = \frac{p}{q}\,u^{\frac{p}{q}-1}\,du = nu^{n-1}\,du.$$

(3) Let n be a negative number $-m$. Then

$$y = u^n = u^{-m} = \frac{1}{u^m}.$$

Since m is positive, we can find $d\,(u^m)$ by the formulas proved above. Therefore, by V,

$$dy = \frac{u^m d(1) - 1\,d(u^m)}{(u^m)^2} = \frac{-mu^{m-1}du}{u^{2m}} = -mu^{-m-1}\,du = nu^{n-1}\,du.$$

Therefore, whether n is an integer or fraction, positive or negative,

$$d\,(u^n) = nu^{n-1}\,du.$$

If the numerator of a fraction is constant, this formula can be used instead of V. Thus

$$d\left(\frac{c}{u}\right) = d\,(cu^{-1}) = -cu^{-2}\,du.$$

Example 1. $y = 4\,x^3$.

Using formulas III and VI,

$$dy = 4\,d\,(x^3) = 4 \cdot 3\,x^2\,dx = 12\,x^2\,dx.$$

Ex. 2. $y = \sqrt{x} + \dfrac{1}{\sqrt{x}} + 3$.

This can be written

$$y = x^{\frac{1}{2}} + x^{-\frac{1}{2}} + 3.$$

Consequently, by II and VI,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d\,(x^{\frac{1}{2}})}{dx} + \frac{d\,(x^{-\frac{1}{2}})}{dx} + \frac{d\,(3)}{dx}\\ &= \frac{1}{2}\,x^{-\frac{1}{2}}\frac{dx}{dx} - \frac{1}{2}\,x^{-\frac{3}{2}}\frac{dx}{dx} + 0\\ &= \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x^3}}.\end{aligned}$$

Ex. 3. $y = (x + a)\,(x^2 - b^2)$.

Using IV, with $u = x + a$, $v = x^2 - b^2$,

$$\begin{aligned}\frac{dy}{dx} &= (x + a)\frac{d}{dx}(x^2 - b^2) + (x^2 - b^2)\frac{d}{dx}(x + a)\\ &= (x + a)\,(2\,x - 0) + (x^2 - b^2)\,(1 + 0)\\ &= 3\,x^2 + 2\,ax - b^2.\end{aligned}$$

Ex. 4. $y = \dfrac{x^2 + 1}{x^2 - 1}$.

Using V, with $u = x^2 + 1$, $v = x^2 - 1$,

$$dy = \frac{(x^2 - 1)\, d\,(x^2 + 1) - (x^2 + 1)\, d\,(x^2 - 1)}{(x^2 - 1)^2}$$

$$= \frac{(x^2 - 1)\, 2\,x\,dx - (x^2 + 1)\, 2\,x\,dx}{(x^2 - 1)^2}$$

$$= - \frac{4\,x\,dx}{(x^2 - 1)^2}.$$

Ex. 5. $y = \sqrt{x^2 - 1}$.

Using VI, with $u = x^2 - 1$,

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 - 1)^{\frac{1}{2}} = \tfrac{1}{2}(x^2 - 1)^{-\frac{1}{2}} \frac{d}{dx}(x^2 - 1)$$

$$= \tfrac{1}{2}(x^2 - 1)^{-\frac{1}{2}}(2\,x) = \frac{x}{\sqrt{x^2 - 1}}.$$

Ex. 6. $x^2 + xy - y^2 = 1$.

We can consider y a function of x determined by the equation. Then

$$d\,(x^2) + d\,(xy) - d\,(y^2) = d\,(1) = 0,$$

that is,

$$2\,x\,dx + x\,dy + y\,dx - 2\,y\,dy = 0,$$

$$(2\,x + y)\,dx + (x - 2\,y)\,dy = 0.$$

Consequently,

$$\frac{dy}{dx} = \frac{2\,x + y}{2\,y - x}.$$

Ex. 7. $x = t + \dfrac{1}{t}, \quad y = t - \dfrac{1}{t}.$

In this case

$$dx = dt - \frac{dt}{t^2}, \quad dy = dt + \frac{dt}{t^2}.$$

Consequently,

$$\frac{dy}{dx} = \frac{1 + \dfrac{1}{t^2}}{1 - \dfrac{1}{t^2}} = \frac{t^2 + 1}{t^2 - 1}.$$

Ex. 8. Find an approximate value of $y = \left(\dfrac{1 - x}{1 + x}\right)^{\frac{1}{3}}$ when $x = 0.2$.

When $x = 0$, $y = 1$. Also

$$dy = -\frac{2\,dx}{3\,(1-x)^{\frac{2}{3}}\,(1+x)^{\frac{4}{3}}}.$$

When $x = 0$ this becomes

$$dy = -\tfrac{2}{3}\,dx.$$

If we assume that dy is approximately equal to Δy, the change in y when x changes from 0 to 0.2 is approximately

$$dy = -\tfrac{2}{3}\,(0.2) = -0.13.$$

The required value is then

$$y = 1 - 0.13 = .87.$$

EXERCISES

In the following exercises show that the differentials and derivatives have the values given:

1. $y = 3\,x^4 + 4\,x^3 - 6\,x^2 + 5, \quad dy = 12\,(x^3 + x^2 - x)\,dx.$

2. $y = 2\,x^{\frac{3}{2}} - 3\,x^{\frac{2}{3}} + 1, \quad \dfrac{dy}{dx} = \dfrac{3\,x^{\frac{5}{6}} - 2}{x^{\frac{1}{3}}}.$

3. $y = \dfrac{x^3 - x^2 + 1}{5}, \quad \dfrac{dy}{dx} = \dfrac{3\,x^2 - 2\,x}{5}.$

4. $y = (x + 2\,a)\,(x - a)^2, \quad dy = 3\,(x^2 - a^2)\,dx.$

5. $y = x\,(2\,x - 1)\,(3\,x + 2), \quad \dfrac{dy}{dx} = 18\,x^2 + 2\,x - 2.$

6. $y = \dfrac{1}{x^2 + 1}, \quad dy = \dfrac{-2\,x\,dx}{(x^2 + 1)^2}.$

7. $y = \dfrac{2\,x + 3}{4\,x - 5}, \quad dy = \dfrac{-22\,dx}{(4\,x - 5)^2}.$

8. $\dfrac{d}{dx}\dfrac{1 - 2\,x}{(x - 1)^2} = \dfrac{2\,x}{(x - 1)^3}.$

9. $d\sqrt{1 + 2t - t^2} = \dfrac{(1 - t)\,dt}{\sqrt{1 + 2t - t^2}}.$

10. $\dfrac{d}{d\theta}\,\dfrac{1}{\theta + \sqrt{\theta^2 - 1}} = \dfrac{\sqrt{\theta^2 - 1} - \theta}{\sqrt{\theta^2 - 1}}.$

11. $\dfrac{d}{ds}\,s\sqrt{a^2 - s^2} = \dfrac{a^2 - 2\,s^2}{\sqrt{a^2 - s^2}}.$

12. $d\,\dfrac{y}{\sqrt{a^2 - y^2}} = \dfrac{a^2\,dy}{(a^2 - y^2)^{\frac{3}{2}}}.$

13. $\dfrac{d}{dx}\left(\dfrac{a}{x}\sqrt{a^2 - x^2}\right) = -\dfrac{a^3}{x^2\sqrt{a^2 - x^2}}.$

14. $\dfrac{d}{dx}\sqrt{\dfrac{x^2 - 1}{x^2 + 1}} = \dfrac{2\,x}{(x^2 + 1)\sqrt{x^4 - 1}}.$

15. $d\,\dfrac{(2 + 3x^6)^{\frac{5}{3}}}{x^{10}} = -\dfrac{20}{x^{11}}\,(2 + 3x^6)^{\frac{2}{3}}\,dx.$

16. $\dfrac{d}{dx}\,x^2y^2 = 2\,xy^2 + 2\,x^2y\,\dfrac{dy}{dx}.$

17. $d\sqrt{x^2 + y^2} = \dfrac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}}.$

18. $y = (x+1)(2-3x)^2(2x-3)^3$,

$$\frac{dy}{dx} = (24 + 13x - 36x^2)(2-3x)(2x-3)^2.$$

19. $y = \dfrac{x^m}{(a+bx^n)^{\frac{m}{n}}}$, $\qquad \dfrac{dy}{dx} = \dfrac{max^{m-1}}{(a+bx^n)^{\frac{m}{n}+1}}$.

20. $y = \dfrac{2x^2-1}{3x^3}\sqrt{x^2+1}$, $\qquad \dfrac{dy}{dx} = \dfrac{1}{x^4\sqrt{x^2+1}}$.

21. $y = \dfrac{(x+\sqrt{1+x^2})^{n+1}}{n+1} + \dfrac{(x+\sqrt{1+x^2})^{n-1}}{n-1}$,

$$dy = 2(x+\sqrt{1+x^2})^n\,dx.$$

22. $x^2 + y^2 = a^2$, $\qquad \dfrac{dy}{dx} = -\dfrac{x}{y}$.

23. $x^3 + y^3 = 3axy$, $\qquad \dfrac{dy}{dx} = \dfrac{ay - x^2}{y^2 - ax}$.

24. $2x^2 - 3xy + 4y^2 = 3x$, $\qquad \dfrac{dy}{dx} = \dfrac{4x - 3y - 3}{3x - 8y}$.

25. $\dfrac{x}{y} + \dfrac{y}{x} = 1$, $\qquad y\,dx - x\,dy = 0$.

26. $y = \dfrac{1}{x}$, $\qquad \dfrac{dx}{\sqrt{1+x^4}} + \dfrac{dy}{\sqrt{1+y^4}} = 0$.

27. $y^{2n} + x^m y^n = x^{2m}$, $\qquad my\,dx = nx\,dy$.

28. $x = \dfrac{t}{t-1}$, $\quad y = \dfrac{2t+3}{t-1}$, $\qquad \dfrac{dy}{dx} = 5$.

29. $x = t - \sqrt{t^2-1}$, $\quad y = t + \sqrt{t^2-1}$, $\qquad x\,dy + y\,dx = 0$.

30. $x = \dfrac{3at}{1+t^3}$, $\quad y = \dfrac{3at^2}{1+t^3}$, $\qquad \dfrac{dy}{dx} = \dfrac{2t - t^4}{1 - 2t^3}$.

31. Given $y = \dfrac{x}{\sqrt{x^2+9}}$, find an approximate value for y when $x = 4.2$.

32. Find an approximate value of

$$\sqrt{\frac{x^2 - x + 1}{x^2 + x + 1}}$$

when $x = .3$.

33. Given $y = x^6$, find dy and Δy when x changes from 3 to 3.1. Is dy a satisfactory approximation for Δy? Express the difference as a percentage of Δy.

34. Find the slope of the curve

$$y = x(x^5 + 31)^{\frac{4}{5}}$$

at the point $x = 1$.

35. Find the points on the parabola $y^2 = 4ax$ where the tangent is inclined at an angle of 45° to the x-axis.

36. Given $y = (a + x)\sqrt{a - x}$, for what values of x does y increase as x increases and for what values does y decrease as x increases?

37. Find the points $P\ (x, y)$ on the curve

$$y = x + \frac{1}{x}$$

where the tangent is perpendicular to the line joining P to the origin.

38. Find the angle at which the circle

$$x^2 + y^2 = 2\,x - 3\,y$$

intersects the x-axis at the origin.

39. A line through the point (1, 2) cuts the x-axis at $(x, 0)$ and the y-axis at $(0, y)$. Find $\dfrac{dy}{dx}$.

40. If $x^2 - x + 2 = 0$, why is the equation

$$\frac{d}{dx}(x^2 - x + 2) = 0$$

not satisfied?

41. The distances x, x' of a point and its image from a lens are connected by the equation

$$\frac{1}{x} + \frac{1}{x'} = \frac{1}{f},$$

f being constant. If L is the length of a small object extending along the axis perpendicular to the lens and L' is the length of its image, show that

$$\frac{L'}{L} = \left(\frac{x'}{x}\right)^2$$

approximately, x and x' being the distances of the object and its image from the lens.

24. Higher Derivatives. — The first derivative $\dfrac{dy}{dx}$ is a function of x. Its derivative with respect to x, written $\dfrac{d^2y}{dx^2}$, is called the second derivative of y with respect to x. That is,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right).$$

Similarly,

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right),$$

$$\frac{d^4y}{dx^4} = \frac{d}{dx}\left(\frac{d^3y}{dx^3}\right), \text{ etc.}$$

The derivatives of $f(x)$ with respect to x are often written $f'(x), f''(x), f'''(x)$, etc. Thus, if $y = f(x)$,

$$\frac{dy}{dx} = f'(x),\quad \frac{d^2y}{dx^2} = f''(x),\quad \frac{d^3y}{dx^3} = f'''(x), \text{ etc.}$$

Example 1. $y = x^3$.

Differentiation with respect to x gives

$$\frac{dy}{dx} = 3x^2,$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(3x^2) = 6x,$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}(6x) = 6,$$

$$\frac{d^4y}{dx^4} = \frac{d}{dx}(6) = 0.$$

All higher derivatives are zero.

Ex. 2. $x^2 + xy + y^2 = 1$.

Differentiating with respect to x,

$$2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0,$$

whence

$$\frac{dy}{dx} = -\frac{2x + y}{x + 2y}.$$

The second derivative is

$$\frac{d^2y}{dx^2} = -\frac{d}{dx}\left(\frac{2x + y}{x + 2y}\right) = \frac{3x\frac{dy}{dx} - 3y}{(x + 2y)^2}.$$

Replacing $\frac{dy}{dx}$ by its value in terms of x and y and reducing,

$$\frac{d^2y}{dx^2} = -\frac{6(x^2 + xy + y^2)}{(x + 2y)^3} = -\frac{6}{(x + 2y)^3}.$$

The last expression is obtained by using the equation of the curve $x^2 + xy + y^2 = 1$. By differentiating this second derivative we could find the third derivative, etc.

25. Change of Variable. — We have represented the second derivative by $\frac{d^2y}{dx^2}$. This can be regarded as the quotient obtained by dividing a second differential

$$d^2y = d\,(dy)$$

by $(dx)^2$. The value of d^2y will however depend on the variable with respect to which y is differentiated.

Thus, suppose $y = x^2$, $x = t^3$. Then $\frac{d^2y}{dx^2} = 2$ and so

$$d^2y = 2\,(dx)^2 = 2\,(3\,t^2\,dt)^2 = 18\,t^4\,(dt)^2.$$

If, however, we differentiate with respect to t, since $y = t^6$, $\frac{d^2y}{dt^2} = 30\,t^4$ and

$$d^2y = 30\,t^4\,(dt)^2,$$

which is not equal to the value obtained when we differentiated y with respect to x.

For this reason we shall not use differentials of the second or higher orders except in the numerators of derivatives. Two derivatives like $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$ must not be combined like fractions because d^2y does not have the same value in the two cases.

If we have derivatives with respect to t and wish to find derivatives with respect to x, they can be found by using the identical relation

$$\frac{d}{dx}\,u = \frac{du}{dt}\,\frac{dt}{dx} = \frac{\frac{du}{dt}}{\frac{dx}{dt}}. \tag{25}$$

For example,

$$\frac{d}{dx}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} = \frac{\frac{d^2y}{dt^2}}{\frac{dx}{dt}}.$$

Example. Given $x = t - \frac{1}{t}$, $y = t + \frac{1}{t}$, find $\frac{d^2y}{dx^2}$.

In this case

$$\frac{dy}{dx} = \frac{dt - \frac{dt}{t^2}}{dt + \frac{dt}{t^2}} = \frac{t^2 - 1}{t^2 + 1}.$$

Consequently,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{t^2 - 1}{t^2 + 1}\right) = \frac{4\,t}{(t^2 + 1)^2}\,\frac{dt}{dx} = \frac{4\,t}{(t^2 + 1)^2}\,\frac{1}{1 + \frac{1}{t^2}} = \frac{4\,t^3}{(t^2 + 1)^3}.$$

EXERCISES

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in each of the following exercises:

1. $y = \frac{x + 1}{x - 1}$.

2. $y = \sqrt{a^2 - x^2}$.

3. $y = (x - 1)^3 (x + 2)^4$.

4. $y^2 = 4\,x$.

5. $x^2 + y^2 = a^2$.

6. $x^2 - 2\,y^2 = 1$.

7. $xy = x + y$.

8. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

9. If a and b are constant and $y = ax^2 + bx$, show that

$$x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + bx = 0.$$

10. If a, b, c, d are constant and $y = ax^3 + bx^2 + cx + d$, show that

$$\frac{d^4y}{dx^4} = 0.$$

11. Show that

$$\frac{d}{dt}\left(t\frac{dx}{dt} - x\right) = t\frac{d^2x}{dt^2}.$$

12. Show that

$$\frac{d}{dx}\left(x^3 \frac{d^3y}{dx^3} - 3\,x^2 \frac{d^2y}{dx^2} + 6\,x \frac{dy}{dx} - 6\,y\right) = x^3 \frac{d^4y}{dx^4}.$$

13. Given $x = t^2 + t^3$, $y = t^2 - t^3$, find $\frac{d^2y}{dx^2}$ and $\frac{d^2x}{dy^2}$.

14. By differentiating the equation

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

with respect to x, find $\frac{d^2y}{dx^2}$ in terms of derivatives of x with respect to y.

CHAPTER IV

RATES

26. Rate of Change. — If the change in a quantity z is proportional to the time in which it occurs, z is said to change at a constant rate. If Δz is the change occurring in an interval of time Δt, the rate of change of z is

$$\frac{\Delta z}{\Delta t}.$$

If the rate of change of z is not constant, it will be nearly constant if the interval Δt is very short. Then $\frac{\Delta z}{\Delta t}$ is approximately the rate of change, the approximation becoming greater as the increments become less. The exact rate of change at the time t is consequently defined as

$$\lim_{\Delta t \doteq 0} \frac{\Delta z}{\Delta t} = \frac{dz}{dt}, \tag{26}$$

that is, *the rate of change of any quantity is its derivative with respect to the time.*

If the quantity is increasing, its rate of change is positive; if decreasing, the rate is negative.

27. Velocity Along a Straight Line. — Let a particle P move along a straight line (Fig. 27). Let $s = OP$ be con-

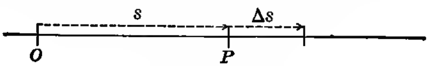

FIG. 27.

sidered positive on one side of O, negative on the other. If the particle moves with constant velocity the distance Δs in the time Δt, its velocity is

$$\frac{\Delta s}{\Delta t}.$$

If the velocity is not constant, it will be nearly so when Δt is very short. Therefore $\frac{\Delta s}{\Delta t}$ is approximately the velocity, the approximation becoming greater as Δt becomes less. The velocity at the time t is therefore defined as

$$v = \lim_{\Delta t \doteq 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}. \tag{27}$$

This equation shows that ds is the distance the particle would move in a time dt if the velocity remained constant. As a rule the velocity will not be constant and so ds will be different from the distance the particle does move in the time dt.

When s is increasing, the velocity is positive; when s is decreasing, the velocity is negative.

Example. A body starting from rest falls approximately

$$s = 16\,t^2$$

feet in t seconds. Find its velocity at the end of 10 seconds.

The velocity at any time t is

$$v = \frac{ds}{dt} = 32\,t \text{ ft./sec.*}$$

At the end of 10 seconds it is

$$v = 320 \text{ ft./sec.}$$

28. Acceleration Along a Straight Line. — The acceleration of a particle moving along a straight line is defined as the rate of change of its velocity. That is

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}. \tag{28}$$

This equation shows that dv is the amount v would increase in the time dt if the acceleration remained constant.

The acceleration is positive when the velocity is increasing, negative when it is decreasing.

Example. At the end of t seconds the vertical height of a ball thrown upward with a velocity of 100 ft./sec. is

$$h = 100\,t - 16\,t^2.$$

Find its velocity and acceleration. Also find when it is rising, when falling, and when it reaches the highest point.

* The notation ft./sec. means feet per second. Similarly, ft./sec.^{2}, used for acceleration, means feet per second per second.

The velocity and acceleration are

$$v = \frac{dh}{dt} = (100 - 32\,t) \text{ ft./sec.},$$

$$a = \frac{dv}{dt} = -32 \text{ ft./sec.}^2.$$

The ball will be rising while v is positive, that is, until $t = \frac{100}{32} = 3\frac{1}{8}$. It will be falling after $t = 3\frac{1}{8}$. It will be at the highest point when $t = 3\frac{1}{8}$.

29. Angular Velocity and Acceleration. — Consider a body rotating about a fixed axis. Let θ be the angle turned through at time t. The *angular velocity* is defined as the rate of change of θ, that is,

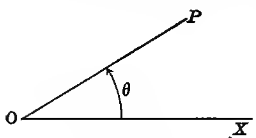

FIG. 29.

$$\text{angular velocity} = \omega = \frac{d\theta}{dt}.$$

The *angular acceleration* is the rate of change of angular velocity, that is,

$$\text{angular acceleration} = \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}.$$

Example 1. A wheel is turning 100 revolutions per minute about its axis. Find its angular velocity.

The angle turned through in one minute will be

$$\omega = 100 \cdot 2\,\pi = 200\,\pi \text{ radians/min.}$$

Ex. 2. A wheel, starting from rest under the action of a constant moment (or twist) about its axis, will turn in t seconds through an angle

$$\theta = kt^2,$$

k being constant. Find its angular velocity and acceleration at time t.

By definition

$$\omega = \frac{d\theta}{dt} = 2\,kt \text{ rad./sec.},$$

$$\alpha = \frac{d\omega}{dt} = 2\,k \text{ rad./sec.}^2.$$

30. Related Rates. — In many cases the rates of change of certain variables are known and the rates of others are to be calculated. This is done by expressing the quantities whose rates are wanted in terms of those whose rates are known and taking the derivatives with respect to t.

Example 1. The radius of a cylinder is increasing 2 ft./sec. and its altitude decreasing 3 ft./sec. Find the rate of change of its volume.

Let r be the radius and h the altitude. Then

$$v = \pi r^2 h.$$

The derivative with respect to t is

$$\frac{dv}{dt} = \pi r^2 \frac{dh}{dt} + 2\,\pi r h \frac{dr}{dt}.$$

By hypothesis

$$\frac{dr}{dt} = 2, \quad \frac{dh}{dt} = -3.$$

Hence

$$\frac{dv}{dt} = 4\,\pi r h - 3\,\pi r^2.$$

This is the rate of increase when the radius is r and altitude h. If $r = 10$ ft. and $h = 6$ ft.,

$$\frac{dv}{dt} = -60\,\pi \text{ cu. ft./sec.}$$

Ex. 2. A ship B sailing south at 16 miles per hour is northwest of a ship A sailing east at 10 miles per hour. At what rate are the ships approaching?

Let x and y be the distances of the ships A and B from the point where their paths cross. The distance between the ships is then

$$s = \sqrt{x^2 + y^2}.$$

This distance is changing at the rate

$$\frac{ds}{dt} = \frac{x\dfrac{dx}{dt} + y\dfrac{dy}{dt}}{\sqrt{x^2 + y^2}}.$$

By hypothesis,

$$\frac{dx}{dt} = 10, \ \frac{dy}{dt} = -16, \ \frac{x}{\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}} = \cos 45° = \frac{1}{\sqrt{2}}.$$

Therefore

$$\frac{ds}{dt} = \frac{10-16}{\sqrt{2}} = -3\sqrt{2} \text{ mi./hr.}$$

The negative sign shows that s is decreasing, that is, the ships are approaching.

EXERCISES

1. From the roof of a house 50 ft. above the street a ball is thrown upward with a speed of 100 ft. per second. Its height above the ground t seconds later will be

$$h = 50 + 100\,t - 16\,t^2.$$

Find its velocity and acceleration when $t = 2$. How long does it continue to rise? What is the highest point reached?

2. A body moves in a straight line according to the law

$$s = \tfrac{1}{4}\,t^4 - 4\,t^3 + 16\,t^2.$$

Find its velocity and acceleration. During what interval is the velocity decreasing? When is it moving backward?

3. If v is the velocity and a the acceleration of a particle moving along the x-axis, show that

$$a\,dx = v\,dv.$$

4. If a particle moves along a line with the velocity

$$v^2 = 2\,gs,$$

where g is constant and s the distance from a fixed point in the line, show that the acceleration is constant.

5. When a particle moves with constant speed around a circle with center at the origin, its shadow on the x-axis moves with velocity v satisfying the equation

$$v^2 + n^2x^2 = n^2r^2,$$

n and r being constant. Show that the acceleration of the shadow is proportional to its distance from the origin.

6. A wheel is turning 500 revolutions per minute. What is its angular velocity? If the wheel is 4 ft. in diameter, with what speed does it drive a belt?

7. A rotating wheel is brought to rest by a brake. Assuming the friction between brake and wheel to be constant, the angle turned through in a time t will be

$$\theta = a + bt - ct^2,$$

a, b, c being constants. Find the angular velocity and acceleration. When will the wheel come to rest?

8. A wheel revolves according to the law $\omega = 30\,t + t^2$, where ω is the speed in radians per minute and t the time since the wheel started. A second wheel turns according to the law $\theta = \frac{1}{4}\,t^2$, where t is the time in seconds and θ the angle in degrees through which it has turned. Which wheel is turning faster at the end of one minute and how much?

9. A wheel of radius r rolls along a line. If v is the velocity and a the acceleration of its center, ω the angular velocity and α the angular acceleration about its axis, show that

$$v = r\omega, \qquad a = r\alpha.$$

10. The depth of water in a cylindrical tank, 6 feet in diameter, is increasing 1 foot per minute. Find the rate at which the water is flowing in.

11. A stone dropped into a pond sends out a series of concentric ripples. If the radius of the outer ripple increases steadily at the rate of 6 ft./sec., how rapidly is the area of disturbed water increasing at the end of 2 seconds?

12. At a certain instant the altitude of a cone is 7 ft. and the radius of its base 3 ft. If the altitude is increasing 2 ft./sec. and the radius of its base decreasing 1 ft./sec., how fast is the volume increasing or decreasing?

13. The top of a ladder 20 feet long slides down a vertical wall. Find the ratio of the speeds of the top and bottom when the ladder makes an angle of 30 degrees with the ground.

14. The cross section of a trough 10 ft. long is an equilateral triangle. If water flows in at the rate of 10 cu. ft./sec., find the rate at which the depth is increasing when the water is 18 inches deep.

15. A man 6 feet tall walks at the rate of 5 feet per second away from a lamp 10 feet from the ground. When he is 20 feet from the lamp post, find the rate at which the end of his shadow is moving and the rate at which his shadow is growing.

16. A boat moving 8 miles per hour is laying a cable. Assuming that the water is 1000 ft. deep, the cable is attached to the bottom and stretches in a straight line to the stern of the boat, at what rate is the cable leaving the boat when 2000 ft. have been paid out?

17. Sand when poured from a height on a level surface forms a cone with constant angle β at the vertex, depending on the material. If the sand is poured at the rate of c cu. ft./sec., at what rate is the radius increasing when it equals a?

18. Two straight railway tracks intersect at an angle of 60 degrees. On one a train is 8 miles from the junction and moving toward it at the rate of 40 miles per hour. On the other a train is 12 miles from the junction and moving from it at the rate of 10 miles per hour. Find the rate at which the trains are approaching or separating.

19. An elevated car running at a constant elevation of 50 ft. above the street passes over a surface car, the tracks crossing at right angles. If the speed of the elevated car is 16 miles per hour and that of the surface car 8 miles, at what rate are the cars separating 10 seconds after they meet?

20. The rays of the sun make an angle of 30 degrees with the horizontal. A ball drops from a height of 64 feet. How fast is its shadow moving just before the ball hits the ground?

CHAPTER V

MAXIMA AND MINIMA

31. A function of x is said to have a *maximum* at $x = a$, if when $x = a$ the function is greater than for any other value in the immediate neighborhood of a. It has a *minimum* if when $x = a$ the function is less than for any other value of x sufficiently near a.

If we represent the function by y and plot the curve $y = f(x)$, a maximum occurs at the top, a minimum at the bottom of a wave.

If the derivative is continuous, as in Fig. 31a, the tangent is horizontal at the highest and lowest points of a wave and the slope is zero. Hence in determining maxima and minima of a function $f(x)$ we first look for values of x such that

$$\frac{d}{dx} f(x) = f'(x) = 0.$$

If a is a root of this equation, $f(a)$ may be a maximum, a minimum, or neither.

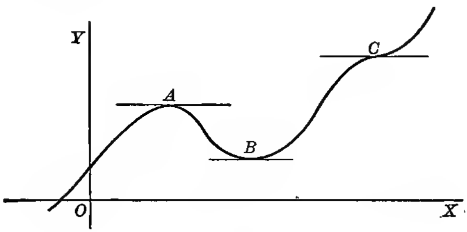

FIG. 31a.

If the slope is positive on the left of the point and negative on the right, as at A, the curve falls on both sides and the ordinate is a maximum. That is, $f(x)$ *has a maximum value* *at $x = a$, if $f'(x)$ is positive for values of x a little less and negative for values a little greater than a.*

If the slope is negative on the left and positive on the right, as at B, the curve rises on both sides and the ordinate is a minimum. That is, *$f(x)$ has a minimum at $x = a$, if $f'(x)$ is negative for values of x a little less and positive for values a little greater than a.*

If the slope has the same sign on both sides, as at C, the curve rises on one side and falls on the other and the ordinate is neither a maximum nor a minimum. That is, *$f(x)$ has neither a maximum nor a minimum at $x = a$ if $f'(x)$ has the same sign on both sides of a.*

Example 1. The sum of two numbers is 5. Find the maximum value of their product.

Let one of the numbers be x. The other is then $5 - x$. The value of x is to be found such that the product

$$y = x(5 - x) = 5x - x^2$$

is a maximum. The derivative is

$$\frac{dy}{dx} = 5 - 2x.$$

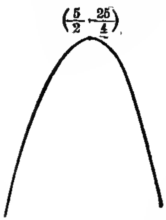

FIG. 31b.

This is zero when $x = \frac{5}{2}$. If x is less than $\frac{5}{2}$, the derivative is positive. If x is greater than $\frac{5}{2}$ the derivative is negative. Near $x = \frac{5}{2}$ the graph then has the shape shown in Fig. 31b. At $x = \frac{5}{2}$ the function has its greatest value

$$\frac{5}{2}\left(5 - \frac{5}{2}\right) = \frac{25}{4}.$$

Ex. 2. Find the shape of the pint cup which requires for its construction the least amount of tin.

Let the radius of base be r and the depth h. The area of tin used is

$$A = \pi r^2 + 2\pi rh.$$

Let v be the number of cubic inches in a pint. Then

$$v = \pi r^2 h.$$

Consequently, $$h = \frac{v}{\pi r^2}$$

and $$A = \pi r^2 + \frac{2v}{r}.$$

Since π and v are constants,

$$\frac{dA}{dr} = 2\pi r - \frac{2v}{r^2} = 2\left(\frac{\pi r^3 - v}{r^2}\right).$$

This is zero if $\pi r^3 = v$. If there is a maximum or minimum it must then occur when

$$r = \sqrt[3]{\frac{v}{\pi}};$$

for, if r has any other value, $\frac{dA}{dr}$ will have the same sign on both sides of that value and A will be neither a maximum nor a minimum. Since the amount of tin used cannot be zero there must be a least amount. This must then be the value of A when $v = \pi r^3$. Also $v = \pi r^2 h$. We therefore conclude that $r = h$. The cup requiring the least tin thus has a depth equal to the radius of its base.

Ex. 3. The strength of a rectangular beam is proportional to the product of its width by the square of its depth. Find the strongest beam that can be cut from a circular log 24 inches in diameter.

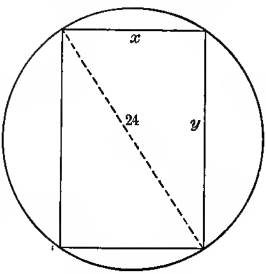

FIG. 31c.

In Fig. 31c is shown a section of the log and beam. Let x be the breadth and y the depth of the beam. Then

$$x^2 + y^2 = (24)^2.$$

The strength of the beam is

$$S = kxy^2 = kx\,(24^2 - x^2),$$

k being constant. The derivative of S is

$$\frac{dS}{dx} = k\,(24^2 - 3\,x^2).$$

If this is zero, $x = \pm 8\sqrt{3}$. Since x is the breadth of the beam, it cannot be negative. Hence

$$x = 8\sqrt{3}$$

is the only solution. Since the log cannot be infinitely strong, there must be a strongest beam. Since no other value can give either a maximum or a minimum, $x = 8\sqrt{3}$ must be the width of the strongest beam. The corresponding depth is $y = 8\sqrt{6}$.

Ex. 4. Find the dimensions of the largest right circular cylinder inscribed in a given right circular cone.

Let r be the radius and h the altitude of the cone. Let x be the radius and y the altitude of an inscribed cylinder (Fig. 31d). From the similar triangles DEC and ABC,

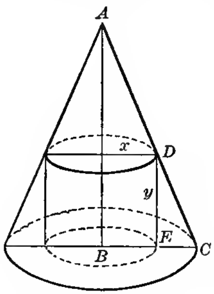

FIG. 31d.

$$\frac{DE}{EC} = \frac{AB}{BC},$$

that is,

$$\frac{y}{r - x} = \frac{h}{r}, \qquad y = \frac{h}{r}\,(r - x).$$

The volume of the cylinder is

$$v = \pi x^2 y = \frac{\pi h}{r}\,(rx^2 - x^3).$$

Equating its derivative to zero, we get

$$2\,rx - 3\,x^2 = 0.$$

Hence $x = 0$ or $x = \frac{2}{3}\,r$. The value $x = 0$ obviously does not give the maximum. Since there is a largest cylinder, its radius must then be $x = \frac{2}{3}\,r$. By substitution its altitude is then found to be $y = \frac{1}{3}\,h$.

32. Method of Finding Maxima and Minima. — The method used in solving these problems involves the following steps:

(1) Decide what is to be a maximum or minimum. Let it be y.

(2) Express y in terms of a *single* variable. Let it be x. It may be convenient to express y temporarily in terms of several variable quantities. If the problem can be solved by our present methods, there will be relations enough to eliminate all but one of these.

(3) Calculate $\frac{dy}{dx}$ and find for what values of x it is zero.

(4) It is usually easy to decide from the problem itself whether the corresponding values of y are maxima or minima. If not, determine the signs of $\frac{dy}{dx}$ when x is a little less and a little greater than the values in question and apply the criteria given in Art. 31.

EXERCISES

Find the maximum and minimum values of the following functions:

1. $2x^2 - 5x + 7$.

2. $6 + 12x - x^3$.

3. $x^4 - 2x^2 + 6$.

4. $\dfrac{x^2}{\sqrt{a^2 - x^2}}$.

Show that the following functions have no maxima or minima:

5. x^3.

6. $x^3 + 4x$.

7. $6x^5 - 15x^4 + 10x^3$.

8. $x\sqrt{a^2 + x^2}$.

9. Show that $x + \frac{1}{x}$ has a maximum and a minimum and that the maximum is less than the minimum.

10. The sum of the square and the reciprocal of a number is a minimum. Find the number.

11. Show that the largest rectangle with a given perimeter is a square.

12. Show that the largest rectangle that can be inscribed in a given circle is a square.

13. Find the altitude of the largest cylinder that can be inscribed in a sphere of radius a.

14. A rectangular box with square base and open at the top is to be made out of a given amount of material. If no allowance is made for thickness of material or waste in construction, what are the dimensions of the largest box that can be made?

15. A cylindrical tin can closed at both ends is to have a given capacity. Show that the amount of tin used will be a minimum when the height equals the diameter.

16. What are the most economical proportions for an open cylindrical water tank if the cost of the sides per square foot is two-thirds the cost of the bottom per square foot?

17. The top, bottom, and lateral surface of a closed tin can are to be cut from rectangles of tin, the scraps being a total loss. Find the most economical proportions for a can of given capacity.

18. Find the volume of the largest right cone that can be generated by revolving a right triangle of hypotenuse 2 ft. about one of its sides.

19. Four successive measurements of a distance gave a_1, a_2, a_3, a_4 as results. By the theory of least squares the most probable value of the distance is that which makes the sum of the squares of the four errors a minimum. What is that value?

20. If the sum of the length and girth of a parcel post package must not exceed 72 inches, find the dimensions of the largest cylindrical jug that can be sent by parcel post.

21. A circular filter paper of radius 6 inches is to be folded into a conical filter. Find the radius of the base of the filter if it has the maximum capacity.

22. Assuming that the intensity of light is inversely proportional to the square of the distance from the source, find the point on the line joining two sources, one of which is twice as intense as the other, at which the illumination is a minimum.

23. The sides of a trough of triangular section are planks 12 inches wide. Find the width at the top if the trough has the maximum capacity.

24. A fence 6 feet high runs parallel to and 5 feet from a wall. Find the shortest ladder that will reach from the ground over the fence to the wall.

25. A log has the form of a frustum of a cone 29 ft. long, the diameters of its ends being 2 ft. and 1 ft. A beam of square section is to be cut from the log. Find its length if the volume of the beam is a maximum.

26. A window has the form of a rectangle surmounted by a semicircle. If the perimeter is 30 ft., find the dimensions so that the greatest amount of light may be admitted.

27. A piece of wire 6 ft. long is to be cut into 6 pieces, two of one length and four of another. The two former are bent into circles which are held in parallel planes and fastened together by the four remaining pieces. The whole forms a model of a right cylinder. Calculate the lengths into which the wire must be divided to produce the cylinder of greatest volume.

28. Among all circular sectors with a given perimeter, find the one which has the greatest area.

29. A ship B is 75 miles due east of a ship A. If B sails west at 12 miles per hour and A south at 9 miles, find when the ships will be closest together.

30. A man on one side of a river $\frac{1}{2}$ mile wide wishes to reach a point on the opposite side 5 miles further along the bank. If he can walk 4 miles an hour and swim 2 miles an hour, find the route he should take to make the trip in the least time.

31. Find the length of the shortest line which will divide an equilateral triangle into parts of equal area.

32. A triangle is inscribed in an oval curve. If the area of the triangle is a maximum, show graphically that the tangents at the vertices of the triangle are parallel to the opposite sides.

33. A and C are points on the same side of a plane mirror. A ray of light passes from A to C by way of a point B on the mirror. Show that the length of the path ABC will be a minimum when the lines AB, CB make equal angles with the perpendicular to the mirror.

34. Let the velocity of light in air be v_1 and in water v_2. The path of a ray of light from a point A in the air to a point C below the surface of the water is bent at B where it enters the water. If θ_1 and θ_2 are the angles made by AB and BC with the perpendicular to the surface, show that the time required for light to pass from A to C will be least if B is so placed that

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2}.$$

35. The cost per hour of propelling a steamer is proportional to the cube of her speed through the water. Find the speed at which a boat should be run against a current of 5 miles per hour in order to make a given trip at least cost.

36. If the cost per hour for fuel required to run a steamer is proportional to the cube of her speed and is \$20 per hour for a speed of 10 knots, and if the other expenses amount to \$100 per hour, find the most economical speed in still water.

33. Other Types of Maxima and Minima. — The method given in Art. 31 is sufficient to determine maxima and minima if the function and its derivative are one-valued and continuous. In Figs. 33a and 33b are shown some types of maxima and minima that do not satisfy these conditions.

At B and C, Fig. 33a, the tangent is vertical and the derivative infinite. At D the slope on the left is different from that on the right. The derivative is discontinuous. At A and E the curve ends. This happens in problems where values beyond a certain range are impossible. According to

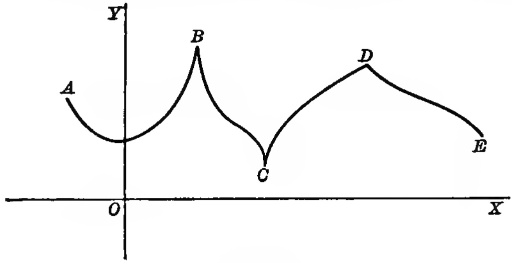

FIG. 33a.

our definition, y has maxima at A, B, D and minima at C and E.

If more than one value of the function corresponds to a single value of the variable, points like A and B, Fig. 33b, may occur. At such points two values of y coincide.

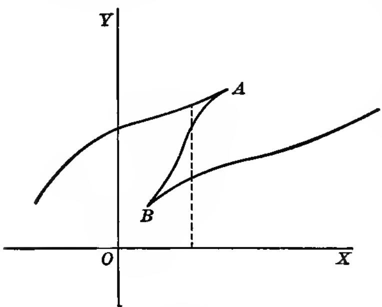

FIG. 33b.

These figures show that in determining maxima and minima special attention must be given to places where the derivative is discontinuous, the function ceases to exist, or two values of the function coincide.

Example 1. Find the maximum and minimum ordinates on the curve $y^3 = x^2$.

In this case, $y = x^{\frac{2}{3}}$ and

$$\frac{dy}{dx} = \frac{2}{3} x^{-\frac{1}{3}}.$$

No finite value of x makes the derivative zero, but $x = 0$

makes it infinite. Since y is never negative, the value 0 is a minimum (Fig. 33c).

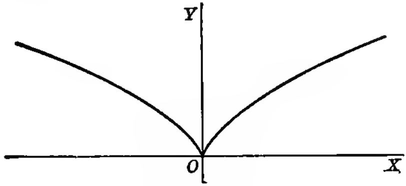

FIG. 33c.

Ex. 2. A man on one side of a river $\frac{1}{2}$ mile wide wishes to reach a point on the opposite side 2 miles down the river. If he can row 6 miles an hour and walk 4, find the route he should take to make the trip in the least time.

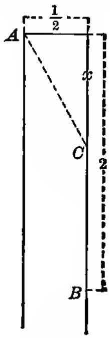

FIG. 33d.

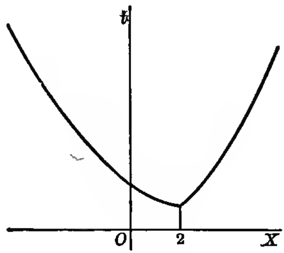

FIG. 33e.

Let A (Fig. 33d) be the starting point and B the destination. Suppose he rows to C, x miles down the river. The time of rowing will be $\frac{1}{6}\sqrt{x^2+\frac{1}{4}}$ and the time of walking $\frac{1}{4}(2-x)$. The total time is then

$$t = \frac{1}{6}\sqrt{x^2+\frac{1}{4}} + \frac{1}{4}(2-x).$$

Equating the derivative to zero, we get

$$\frac{x}{6\sqrt{x^2+\frac{1}{4}}} - \frac{1}{4} = 0,$$

which reduces to $5x^2 + \frac{9}{4} = 0$. This has no real solution.

The trouble is that $\frac{1}{4}(2 - x)$ is the time of walking only if C is above B. If C is below B, the time is $\frac{1}{4}(x - 2)$. The complete value for t is then

$$t = \tfrac{1}{6}\sqrt{x^2 + \tfrac{1}{4}} \pm \tfrac{1}{4}(2 - x),$$

the sign being $+$ if $x < 2$ and $-$ if $x > 2$. The graph of the equation connecting x and t is shown in Fig. 33e. At $x = 2$ the derivative is discontinuous. Since he rows faster than he walks, the minimum obviously occurs when he rows all the way, that is, $x = 2$.

EXERCISES

Find the maximum and minimum values of y on the following curves:

1. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

2. $y^5 = x^2(x - 1)$.

3. $y^3 = x^4 - 1$.

4. $x = t^2 + t^3,\ y = t^2 - t^3$.

5. Find the rectangle of least area having a given perimeter.

6. Find the point on the parabola $y^2 = 4x$ nearest the point $(-1, 0)$.

7. A wire of length l is cut into two pieces, one of which is bent to form a circle, the other a square. Find the lengths of the pieces when the sum of the areas of the square and circle is greatest.

8. Find a point P on the line segment AB such that $PA^2 + PB^2$ is a maximum.

9. If the work per hour of moving a car along a horizontal track is proportional to the square of the velocity, what is the least work required to move the car one mile?

10. If 120 cells of electromotive force E volts and internal resistance 2 ohms are arranged in parallel rows with x cells in series in each row, the current which the resulting battery will send through an external resistance of $\frac{1}{3}$ ohm is

$$C = \frac{60\,xE}{x^2 + 20}.$$

How many cells should be placed in each row to give the maximum current?

CHAPTER VI

DIFFERENTIATION OF TRANSCENDENTAL FUNCTIONS

34. Formulas for Differentiating Trigonometric Functions. — Let u be the *circular measure* of an angle.

$$\begin{array}{rll} \text{VII.} & d\sin u = & \cos u\,du. \\ \text{VIII.} & d\cos u = & -\sin u\,du. \\ \text{IX.} & d\tan u = & \sec^2 u\,du. \\ \text{X.} & d\cot u = & -\csc^2 u\,du. \\ \text{XI.} & d\sec u = & \sec u\tan u\,du. \\ \text{XII.} & d\csc u = & -\csc u\cot u\,du. \end{array}$$

The negative sign occurs in the differentials of all co-functions.

35. The Sine of a Small Angle. — Inspection of a table of natural sines will show that the sine of a small angle is very nearly equal to the circular measure of the angle. Thus

$$\sin 1^\circ = 0.017452,$$

$$\frac{\pi}{180} = 0.017453.$$

We should then expect that

$$\lim_{\theta \doteq 0} \frac{\sin\theta}{\theta} = 1. \qquad (35)$$

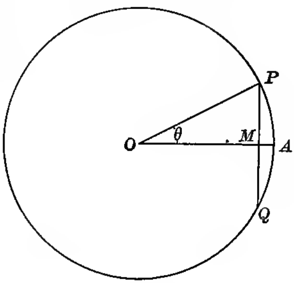

Fig. 35.

To show this graphically, let $\theta = AOP$ (Fig. 35). Draw PM perpendicular to OA. The circular measure of the angle is defined by the equation

$$\theta = \frac{\text{arc}}{\text{rad.}} = \frac{\text{arc } AP}{OP}.$$

Also $\sin\theta = \frac{MP}{OP}$. Hence

$$\frac{\sin\theta}{\theta} = \frac{MP}{\text{arc } AP} = \frac{\text{chord } QP}{\text{arc } QP}.$$

As θ approaches zero, the ratio of the arc to the chord approaches 1 (Art. 53). Therefore the limit of $\frac{\sin\theta}{\theta}$ is 1.

36. Proof of VII, the Differential of the Sine. — Let

$$y = \sin u.$$

Then

$$y + \Delta y = \sin(u + \Delta u)$$

and so

$$\Delta y = \sin(u + \Delta u) - \sin u.$$

It is shown in trigonometry that

$$\sin A - \sin B = 2\cos\tfrac{1}{2}(A + B)\sin\tfrac{1}{2}(A - B).$$

If then $A = u + \Delta u$, $B = u$,

$$\Delta y = 2\cos(u + \tfrac{1}{2}\Delta u)\sin\tfrac{1}{2}\Delta u,$$

whence

$$\frac{\Delta y}{\Delta u} = 2\cos(u + \tfrac{1}{2}\Delta u)\frac{\sin\frac{1}{2}\Delta u}{\Delta u} = \cos(u + \tfrac{1}{2}\Delta u)\frac{\sin\frac{1}{2}\Delta u}{\frac{1}{2}\Delta u}.$$

As Δu approaches zero

$$\frac{\sin\frac{1}{2}\Delta u}{\frac{1}{2}\Delta u} = \frac{\sin\theta}{\theta}$$

approaches 1 and $\cos(u + \frac{1}{2}\Delta u)$ approaches $\cos u$. Therefore

$$\frac{dy}{du} = \cos u.$$

Consequently,

$$dy = \cos u\, du.$$

37. Proof of VIII, the Differential of the Cosine. — By trigonometry

$$\cos u = \sin\left(\frac{\pi}{2} - u\right).$$

Using the formula just proved,

$$d \cos u = d \sin\left(\frac{\pi}{2} - u\right) = \cos\left(\frac{\pi}{2} - u\right) d\left(\frac{\pi}{2} - u\right) = -\sin u\, du.$$

38. Proof of IX, X, XI, and XII. — Differentiating both sides of the equation

$$\tan u = \frac{\sin u}{\cos u}$$

and using the formulas just proved for the differentials of $\sin u$ and $\cos u$,

$$d \tan u = \frac{\cos u\, d \sin u - \sin u\, d \cos u}{\cos^2 u} = \frac{\cos^2 u\, du + \sin^2 u\, du}{\cos^2 u}$$
$$= \sec^2 u\, du.$$

By differentiating both sides of the equations

$$\cot u = \frac{\cos u}{\sin u}, \qquad \sec u = \frac{1}{\cos u}, \qquad \csc u = \frac{1}{\sin u},$$

and using the formulas for the differentials of $\sin u$ and $\cos u$, we obtain the differentials of $\cot u$, $\sec u$ and $\csc u$.

Example 1. $y = \sin^2 (x^2 + 3)$.

Since

$$\sin^2 (x^2 + 3) = [\sin (x^2 + 3)]^2,$$

we use the formula for u^2 and so get

$$\begin{aligned} dy &= 2 \sin (x^2 + 3)\, d \sin (x^2 + 3) \\ &= 2 \sin (x^2 + 3) \cos (x^2 + 3)\, d (x^2 + 3) \\ &= 4\, x \sin (x^2 + 3) \cos (x^2 + 3)\, dx. \end{aligned}$$

Ex. 2. $y = \sec 2\, x \tan 2\, x$.

$$\begin{aligned} \frac{dy}{dx} &= \sec 2\, x \frac{d}{dx} \tan 2\, x + \tan 2\, x \frac{d}{dx} \sec 2\, x \\ &= \sec 2\, x \sec^2 2\, x\, (2) + \tan 2\, x \sec 2\, x \tan 2\, x\, (2) \\ &= 2 \sec 2\, x\, (\sec^2 2\, x + \tan^2 2\, x). \end{aligned}$$

EXERCISES

In the following exercises show that the derivatives and differentials have the values given:

1. $y = 2 \sin 3\, x + 3 \cos 2\, x, \qquad \dfrac{dy}{dx} = 6 (\cos 3\, x - \sin 2\, x).$

2. $y = \sin^2 \dfrac{x}{2}$, $\qquad dy = \sin \dfrac{x}{2} \cos \dfrac{x}{2}\, dx.$

3. $y = 2\cos x \sin 2x - \sin x \cos 2x$, $\qquad dy = 3 \cos x \cos 2x\, dx.$

4. $y = \dfrac{1 - \cos \frac{1}{3}x}{\sin \frac{1}{3}x}$, $\qquad \dfrac{dy}{dx} = \dfrac{1 - \cos \frac{1}{3}x}{3 \sin^2 \frac{1}{3}x}.$

5. $y = \tan 2x + \sec 2x$, $\qquad \dfrac{dy}{dx} = 2 \sec 2x (\sec 2x + \tan 2x).$

6. $y = \cot^2 \dfrac{x}{2} \csc^2 \dfrac{x}{2}$, $\qquad \dfrac{dy}{dx} = -\cot \dfrac{x}{2} \csc^2 \dfrac{x}{2} \left(\csc^2 \dfrac{x}{2} + \cot^2 \dfrac{x}{2}\right).$

7. $x = a \cos t,\ y = a \sin^3 t$, $\qquad \dfrac{dy}{dx} = -3 \sin t \cos t.$

8. $x = a(\phi - \sin\phi),\ y = a(1 - \cos\phi)$, $\qquad \dfrac{dy}{dx} = \cot \dfrac{\phi}{2}.$

9. $x = \cos t + t \sin t,\ y = \sin t - t \cos t$, $\qquad \dfrac{d^2y}{dx^2} = \dfrac{1}{t \cos^3 t}.$

10. $y = \frac{1}{5}\cot^5 x - \frac{1}{3}\cot^3 x + \cot x + x$, $\qquad dy = -\cot^6 x\, dx.$

11. $y = \frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + \tan x$, $\qquad dy = \sec^6 x\, dx.$

12. $u = \frac{1}{7}\sec^7\theta - \frac{2}{5}\sec^5\theta + \frac{1}{3}\sec^3\theta$, $\qquad du = \tan^5\theta \sec^3\theta\, d\theta.$

13. $y = x\left(\dfrac{\cos^3 x}{3} - \cos x\right) + \dfrac{1}{9}\sin^3 x + \dfrac{2}{3}\sin x$, $\qquad dy = x \sin^3 x\, dx.$

14. $y = -\dfrac{\cos x}{2}\left(\dfrac{1}{3}\sin^5 x + \dfrac{5}{12}\sin^3 x + \dfrac{5}{8}\sin x\right) + \dfrac{5}{16}x$, $\qquad dy = \sin^6 x\, dx.$

15. $y = \dfrac{1 + \sin x}{1 - \sin x}$, $\qquad \dfrac{dy}{dx} = \dfrac{2 \cos x}{(1 - \sin x)^2}.$

16. $y = \dfrac{\sec x - \tan x}{\sec x + \tan x}$, $\qquad \dfrac{dy}{dx} = \dfrac{2 \sec x (\tan x - \sec x)}{\sec x + \tan x}.$

17. $y = (\cot x - 3 \tan x)\sqrt{\cot x}$, $\qquad \dfrac{dy}{dx} = -\dfrac{3 \csc^4 x}{2 \cot^{\frac{3}{2}} x}.$

18. If $y = A \cos(nx) + B \sin(nx)$, where A and B are constant, show that

$$\frac{d^2y}{dx^2} + n^2 y = 0.$$

19. Find a constant A such that $y = A \sin 2x$ satisfies the equation

$$\frac{d^2y}{dx^2} + 5y = 3 \sin 2x.$$

20. Find constants A and B such that $y = A \sin 6x + B \cos 6x$ satisfies the equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 8y = 5 \sin 6x.$$

21. Find the slope of the curve $y = 2 \sin x + 3 \cos x$ at the point $x = \dfrac{\pi}{6}$.

22. Find the points on the curve $y = x + \sin 2x$ where the tangent is parallel to the line $y = 2x + 3$.

23. A weight supported by a spring hangs at rest at the origin. If the weight is lifted a distance A and let fall, its height at any subsequent time t will be

$$y = A \cos (2\pi nt),$$

n being constant. Find its velocity and acceleration as it passes the origin. Where is the velocity greatest? Where is the acceleration greatest?

24. A revolving light 5 miles from a straight shore makes one complete revolution per minute. Find the velocity along the shore of the beam of light when it makes an angle of 60 degrees with the shore line.

25. In Ex. 24 with what velocity would the light be rotating if the spot of light is moving along the shore 15 miles per hour when the beam makes with the shore line an angle of 60 degrees?

26. Given that two sides and the included angle of a triangle have at a certain instant the values 6 ft., 10 ft., and 30 degrees respectively, and that these quantities are changing at the rates of 3 ft., -2 ft., and 10 degrees per second, how fast is the area of the triangle changing?

27. OA is a crank and AB a connecting rod attached to a piston B moving along a line through O. If OA revolves about O with angular velocity ω, prove that the velocity of B is ωOC, where C is the point in which the line BA cuts the line through O perpendicular to OB.

28. An alley 8 ft. wide runs perpendicular to a street 27 ft. wide. What is the longest beam that can be moved horizontally along the street into the alley?

29. A needle rests with one end in a smooth hemispherical bowl. The needle will sink to a position in which the center is as low as possible. If the length of the needle equals the diameter of the bowl, what will be the position of equilibrium?

30. A rope with a ring at one end is looped over two pegs in the same horizontal line and held taut by a weight fastened to the free end. If the rope slips freely, the weight will descend as far as possible. Find the angle formed at the bottom of the loop.

31. Find the angle at the bottom of the loop in Ex. 30 if the rope is looped over a circular pulley instead of the two pegs.

32. A gutter is to be made by bending into shape a strip of copper so that the cross section shall be an arc of a circle. If the width of the strip is a, find the radius of the cross section when the carrying capacity is a maximum.

33. A spoke in the front wheel of a bicycle is at a certain instant perpendicular to one in the rear wheel. If the bicycle rolls straight ahead, in what position will the outer ends of the two spokes be closest together?

39. Inverse Trigonometric Functions. — The symbol $\sin^{-1} x$ is used to represent the angle whose sine is x. Thus

$$y = \sin^{-1} x, \qquad x = \sin y$$

are equivalent equations. Similar definitions apply to $\cos^{-1} x$ $\tan^{-1} x$, $\cot^{-1} x$, $\sec^{-1} x$, and $\csc^{-1} x$.

Since supplementary angles and those differing by multiples of 2π have the same sine, an indefinite number of angles are represented by the same symbol $\sin^{-1} x$. The algebraic sign of the derivative depends on the angle differentiated. In the formulas given below it is assumed that $\sin^{-1} u$ and $\csc^{-1} u$ are angles in the first or fourth quadrant, $\cos^{-1} u$ and $\sec^{-1} u$ angles in the first or second quadrant. If angles in other quadrants are differentiated, the opposite sign must be used. The formulas for $\tan^{-1} u$ and $\cot^{-1} u$ are valid in all quadrants.

40. Formulas for Differentiating Inverse Trigonometric Functions. —

$$\text{XIII.} \quad d \sin^{-1} u = \frac{du}{\sqrt{1-u^2}}.$$

$$\text{XIV.} \quad d \cos^{-1} u = -\frac{du}{\sqrt{1-u^2}}.$$

$$\text{XV.} \quad d \tan^{-1} u = \frac{du}{1+u^2}.$$

$$\text{XVI.} \quad d \cot^{-1} u = -\frac{du}{1+u^2}.$$

$$\text{XVII.} \quad d \sec^{-1} u = \frac{du}{u\sqrt{u^2-1}}.$$

$$\text{XVIII.} \quad d \csc^{-1} u = -\frac{du}{u\sqrt{u^2-1}}.$$

41. Proof of the Formulas. — Let

$$y = \sin^{-1} u.$$

Then

$$\sin y = u.$$

Differentiation gives

$$\cos y \, dy = du,$$

whence
$$dy = \frac{du}{\cos y}.$$
But
$$\cos y = \pm\sqrt{1 - \sin^2 y} = \pm\sqrt{1 - u^2}.$$
If y is an angle in the first or fourth quadrant, $\cos y$ is positive. Hence
$$\cos y = \sqrt{1 - u^2}$$
and so
$$dy = \frac{du}{\sqrt{1 - u^2}}.$$

The other formulas are proved in a similar way.

EXERCISES

1. $y = \sin^{-1}(3x - 1)$, $\qquad dy = \dfrac{3\,dx}{\sqrt{6x - 9x^2}}.$

2. $y = \cos^{-1}\left(1 - \dfrac{x}{a}\right)$, $\qquad dy = \dfrac{dx}{\sqrt{2ax - x^2}}.$

3. $y = \tan^{-1}\dfrac{3x}{2}$, $\qquad dy = \dfrac{6\,dx}{9x^2 + 4}.$

4. $y = \cot^{-1}\left(\dfrac{x}{2} - \dfrac{1}{2x}\right)$, $\qquad \dfrac{dy}{dx} = \dfrac{-2}{x^2 + 1}.$

5. $y = \sec^{-1}\sqrt{4x + 1}$, $\qquad dy = \dfrac{dx}{(4x + 1)\sqrt{x}}.$

6. $y = \dfrac{1}{2}\csc^{-1}\dfrac{3}{4x - 1}$, $\qquad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{2 + 2x - 4x^2}}.$

7. $y = \tan^{-1}\dfrac{x - a}{x + a}$, $\qquad \dfrac{dy}{dx} = \dfrac{a}{x^2 + a^2}.$

8. $x = \csc^{-1}(\sec\theta)$, $\qquad \dfrac{dx}{d\theta} = -1.$

9. $y = \sin^{-1}\dfrac{x}{\sqrt{a^2 - x^2}}$, $\qquad \dfrac{dy}{dx} = \dfrac{a^2}{(a^2 - x^2)\sqrt{a^2 - 2x^2}}.$

10. $y = \sec^{-1}\dfrac{1}{\sqrt{1 - x^2}}$, $\qquad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 - x^2}}.$

11. $y = \dfrac{x}{2}\sqrt{a^2 - x^2} + \dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a}$, $\qquad \dfrac{dy}{dx} = \sqrt{a^2 - x^2}.$

12. $y = \tan^{-1}\dfrac{4\sin x}{3 + 5\cos x}$, $\qquad \dfrac{dy}{dx} = \dfrac{4}{5 + 3\cos x}.$

13. $y = \sec^{-1}\dfrac{1}{2x^2 - 1}$, $\qquad \dfrac{dy}{dx} = -\dfrac{2}{\sqrt{1 - x^2}}.$

14. $y = a \sin^{-1}\frac{x}{a} + \sqrt{a^2 - x^2}, \qquad \frac{dy}{dx} = \sqrt{\frac{a-x}{a+x}}.$

15. $y = 2(3x+1)^{\frac{1}{6}} + 4\cot^{-1}\frac{(3x+1)^{\frac{1}{6}}}{2},$

$$\frac{dy}{dx} = \frac{1}{(3x+1)^{\frac{5}{6}} + 4(3x+1)^{\frac{1}{2}}}.$$

16. $y = \frac{1}{6}\tan^{-1}\frac{3x}{2+2x^2}, \qquad \frac{dy}{dx} = \frac{1-x^2}{4x^4+17x^2+4}.$

17. $y = \cos^{-1}\frac{x+1}{2} - \frac{2}{\sqrt{3}}\cos^{-1}\frac{2x}{3-x}, \quad \frac{dy}{dx} = \frac{x+1}{(3-x)\sqrt{3-2x-x^2}}.$

18. $y = \frac{\sqrt{x^2-a^2}}{2a^2x^2} - \frac{1}{2a^3}\csc^{-1}\frac{x}{a}, \quad \frac{dy}{dx} = \frac{1}{x^3\sqrt{x^2-a^2}}.$

19. $y = \tan^{-1}\frac{x+\sqrt{x^2+4x-4}}{2}, \quad \frac{dy}{dx} = \frac{1}{x\sqrt{x^2+4x-4}}.$

20. $y = x\sin^{-1}x + \sqrt{1-x^2}, \qquad \frac{d^2y}{dx^2} = \frac{1}{\sqrt{1-x^2}}.$

21. $y = x^2\sec^{-1}\frac{x}{2} - 2\sqrt{x^2-4}, \qquad \frac{dy}{dx} = 2x\sec^{-1}\frac{x}{2}.$

22. Let s be the arc from the x-axis to the point (x, y) on the circle $x^2 + y^2 = a^2$. Show that

$$\frac{ds}{dx} = -\frac{a}{y}, \quad \frac{ds}{dy} = \frac{a}{x}, \quad ds^2 = dx^2 + dy^2.$$

23. Let A be the area bounded by the circle $x^2 + y^2 = a^2$, the y-axis and the vertical line through (x, y). Show that

$$A = xy + a^2\tan^{-1}\frac{x}{y}, \quad dA = 2y\,dx.$$

24. The end of a string wound on a pulley moves with velocity v along a line perpendicular to the axis of the pulley. Find the angular velocity with which the pulley turns.

25. A tablet 8 ft. high is placed on a wall with its base 20 ft. above the level of an observer's eye. How far from the wall should the observer stand that the angle of vision subtended by the tablet be a maximum?

42. Exponential and Logarithmic Functions. — If a is a positive constant and u a variable, a^u is called an exponential function. If u is a fraction, it is understood that a^u is the positive root.

If $y = a^u$, then u is called the logarithm of y to base a. That is,

$$y = a^u, \qquad u = \log_a y \tag{42a}$$

are by definition equivalent equations. Elimination of u gives the important identity

$$a^{\log_a y} = y. \tag{42b}$$

This expresses symbolically that the logarithm is the power to which the base must be raised to equal the number.

43. The Curves $y = a^x$. — Let a be greater than 1. The graph of

$$y = a^x$$

has the general appearance of Fig. 43. If x receives a small increment h, the increment in y is

$$\Delta y = a^{x+h} - a^x = a^x\,(a^h - 1).$$

This increases as x increases. If then x increases by successive amounts h, the increments in y form steps of increasing height. The curve is thus concave upward and the arc lies below its chord.

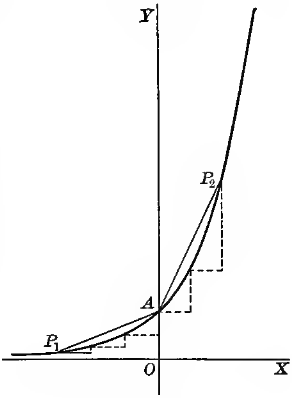

FIG. 43.

The slope of the chord AP joining A (0, 1) and P (x, y) is

$$\frac{a^x - 1}{x}.$$

As P_1 moves toward A the slope of AP_1 increases. As P_2 moves toward A the slope of AP_2 decreases. Furthermore the slopes of AP_2 and AP_1 approach equality; for

$$\frac{a^{-k} - 1}{-k} = a^{-k}\left(\frac{a^k - 1}{k}\right),$$

and a^{-k} approaches 1 when k approaches zero. Now if two numbers, one always increasing, the other always decreasing, approach equality, they approach a common limit. Call this limit m. Then

$$\lim_{x \doteq 0} \frac{a^x - 1}{x} = m. \tag{43}$$

This is the slope of the curve $y = a^x$ at the point where it crosses the y-axis.

44. Definition of e. — We shall now show that there is a number e such that

$$\lim_{x \doteq 0} \frac{e^x - 1}{x} = 1. \tag{44}$$

In fact, let

$$e = a^{\frac{1}{m}}$$

where m is the slope found in Art. 43. Then

$$\lim_{x \doteq 0} \frac{e^x - 1}{x} = \lim_{x \doteq 0} \frac{a^{\frac{x}{m}} - 1}{x} = \frac{1}{m} \lim_{x \doteq 0} \frac{a^{\frac{x}{m}} - 1}{\frac{x}{m}} = \frac{1}{m} \cdot m = 1.$$

The curves $y = a^x$ all pass through the point A (0, 1). Equation (44) expresses that when $a = e$ the slope of the curve at A is 1. If $a > e$ the slope is greater than 1. If $a < e$, the slope is less than 1.

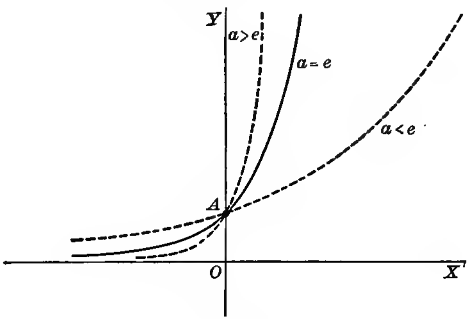

FIG. 44.

We shall find later that

$$e = 2.7183$$

approximately. Logarithms to base e are called *natural* logarithms. In this book we shall represent natural logarithms by the abbreviation ln. Thus ln u means the natural logarithm of u.

45. Differentials of Exponential and Logarithmic Functions. —

XIX. $$de^u = e^u\,du.$$

XX. $$da^u = a^u \ln a\,du.$$

XXI. $$d \ln u = \frac{du}{u}.$$

XXII. $$d \log_a u = \frac{\log_a e\,du}{u}.$$

46. Proof of XIX, the Differential of e^u. — Let

$$y = e^u.$$

Then

$$y + \Delta y = e^{u+\Delta u},$$

whence

$$\Delta y = e^{u+\Delta u} - e^u = e^u\,(e^{\Delta u} - 1)$$

and

$$\frac{\Delta y}{\Delta u} = e^u\,\frac{(e^{\Delta u} - 1)}{\Delta u}.$$

As Δu approaches zero, by (44),

$$\frac{e^{\Delta u} - 1}{\Delta u}$$

approaches 1. Consequently,

$$\frac{dy}{du} = e^u, \qquad dy = e^u\,du.$$

47. Proof of XX, the Differential of a^u. — The identity

$$a = e^{\ln a}$$

gives

$$a^u = e^{u \ln a}.$$

Consequently,

$$da^u = e^{u \ln a}\,d\,(u \ln a) = e^{u \ln a} \ln a\,du = a^u \ln a\,du.$$

48. Proof of XXI and XXII, the Differential of a Logarithm. — Let

$$y = \ln u.$$

Then

$$e^y = u.$$

Differentiating,

$$e^y\,dy = du.$$

Therefore

$$dy = \frac{du}{e^y} = \frac{du}{u}.$$

The derivative of $\log_a u$ is found in a similar way.

Example 1. $y = \ln(\sec^2 x)$.

$$dy = \frac{d \sec^2 x}{\sec^2 x} = \frac{2 \sec x (\sec x \tan x \, dx)}{\sec^2 x} = 2 \tan x \, dx.$$

Ex. 2. $y = 2^{\tan^{-1} x}$.

$$dy = 2^{\tan^{-1} x} \ln 2 \, d(\tan^{-1} x) = \frac{2^{\tan^{-1} x} \ln 2 \, dx}{1 + x^2}.$$

EXERCISES

1. $y = e^{\frac{1}{x}}$, $\quad \dfrac{dy}{dx} = -\dfrac{1}{x^2} e^{\frac{1}{x}}$.

2. $y = a^{\tan 2x}$, $\quad \dfrac{dy}{dx} = 2 a^{\tan 2x} \ln a \sec^2 2x$.

3. $y = e^{\frac{x-1}{x+1}}$, $\quad \dfrac{dy}{dx} = \dfrac{2}{(x+1)^2} e^{\frac{x-1}{x+1}}$.

4. $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$, $\quad \dfrac{dy}{dx} = \left(\dfrac{2}{e^x + e^{-x}}\right)^2$.

5. $y = x^n + n^x$, $\quad \dfrac{dy}{dx} = nx^{n-1} + n^x \ln n$.

6. $y = a^x x^a$, $\quad \dfrac{dy}{dx} = a^x x^{a-1} (a + x \ln a)$.

7. $y = \ln(3x^2 + 5x + 1)$, $\quad \dfrac{dy}{dx} = \dfrac{6x + 5}{3x^2 + 5x + 1}$.

8. $y = \ln \sec^2 x$, $\quad \dfrac{dy}{dx} = 2 \tan x$.

9. $y = \ln\left(x + \sqrt{x^2 - a^2}\right)$, $\quad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{x^2 - a^2}}$.

10. $y = \ln(\sec ax + \tan ax)$, $\quad \dfrac{dy}{dx} = a \sec ax$.

11. $y = \ln(a^x + b^x)$, $\quad \dfrac{dy}{dx} = \dfrac{a^x \ln a + b^x \ln b}{a^x + b^x}$.

12. $y = \ln \sin x + \frac{1}{2} \cos^2 x$, $\quad \dfrac{dy}{dx} = \dfrac{\cos^3 x}{\sin x}$.

13. $y = \dfrac{1}{2} \ln \tan \dfrac{x}{2} - \dfrac{1}{2} \dfrac{\cos x}{\sin^2 x}$, $\quad \dfrac{dy}{dx} = \dfrac{1}{\sin^3 x}$.

14. $y = \frac{1}{4}\ln\frac{x^2}{x^2-4} - \frac{1}{x^2-4}$, $\quad \frac{dy}{dx} = \frac{8}{x(x^2-4)^2}$.

15. $y = \frac{1}{a}\ln\frac{x}{a+\sqrt{a^2-x^2}}$, $\quad \frac{dy}{dx} = \frac{1}{x\sqrt{a^2-x^2}}$.

16. $y = \ln(\sqrt{x+3}+\sqrt{x+2})+\sqrt{(x+3)(x+2)}$, $\quad \frac{dy}{dx} = \sqrt{\frac{x+3}{x+2}}$.

17. $y = \ln(\sqrt{x+a}+\sqrt{x})$, $\quad \frac{dy}{dx} = \frac{1}{2\sqrt{x^2+ax}}$.

18. $y = x\tan^{-1}\frac{x}{a} - \frac{a}{2}\ln(x^2+a^2)$, $\quad \frac{dy}{dx} = \tan^{-1}\frac{x}{a}$.

19. $y = e^{ax}(\sin ax - \cos ax)$, $\quad \frac{dy}{dx} = 2\,ae^{ax}\sin ax$.

20. $y = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln\cos x$, $\quad \frac{dy}{dx} = \tan^5 x$.

21. $x = a\ln t,\ y = \frac{a}{2}\left(t+\frac{1}{t}\right)$, $\quad \frac{dy}{dx} = \frac{1}{2}\left(t-\frac{1}{t}\right)$.

22. $x = e^t + e^{-t},\ y = e^t - e^{-t}$, $\quad \frac{d^2y}{dx^2} = -\frac{4}{y^3}$.

23. $y = \frac{1}{x}\ln x$, $\quad \frac{d^2y}{dx^2} = \frac{1}{x^3}(2\ln x - 3)$.

24. $y = xe^x$, $\quad \frac{d^ny}{dx^n} = (x+n)e^x$.

25. By taking logarithms of both sides of the equation $y = x^n$ and differentiating, show that the formula

$$\frac{d}{dx}x^n = nx^{n-1}$$

is true even when n is irrational.

26. Find the slope of the catenary

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$$

at $x = 0$.

27. Find the points on the curve $y = e^{2x}\sin x$ where the tangent is parallel to the x-axis.

28. If $y = Ae^{nx} + Be^{-nx}$, where A and B are constant, show that

$$\frac{d^2y}{dx^2} - n^2y = 0.$$

29. If $y = ze^{-3x}$, where z is any function of x, show that

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = e^{-3x}\frac{d^2z}{dx^2}.$$

30. For what values of x does

$$y = 5\ln(x-2) + 3\ln(x+2) + 4x$$

increase as x increases?

31. From equation (44) show that

$$e = \lim_{x \doteq 0} (1 + x)^{\frac{1}{x}}.$$

32. If the space described by a point is $s = ae^t + be^{-t}$, show that the acceleration is equal to the space passed over.

33. Assuming the resistance encountered by a body sinking in water to be proportional to the velocity, the distance it descends in a time t is

$$s = \frac{g}{k}t + \frac{g}{k^2}(e^{-kt} - 1),$$

g and k being constants. Show that the velocity v and acceleration a satisfy the equation

$$a = g - kv.$$

Also show that for large values of t the velocity is approximately constant.

34. Assuming the resistance of air proportional to the square of the velocity, a body starting from rest will fall a distance

$$s = \frac{g}{k^2} \ln\left(\frac{e^{kt} + e^{-kt}}{2}\right)$$

in a time t. Show that the velocity and acceleration satisfy the equation

$$a = g - \frac{k^2v^2}{g}.$$

Also show that the velocity approaches a constant value.

CHAPTER VII

GEOMETRICAL APPLICATIONS

49. Tangent Line and Normal. — Let m_1 be the slope of a given curve at $P_1(x_1, y_1)$. It is shown in analytic geometry that a line through (x_1, y_1) with slope m_1 is represented by the equation

$$y - y_1 = m_1(x - x_1).$$

This equation then represents the tangent at (x_1, y_1) where the slope of the curve is m_1.

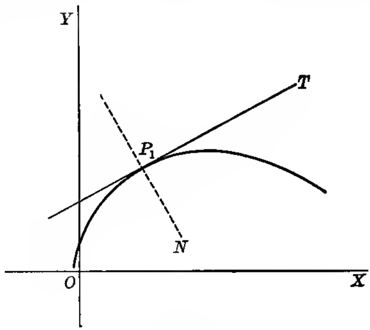

FIG. 49.

The line P_1N perpendicular to the tangent at its point of contact is called the *normal* to the curve at P_1. Since the slope of the tangent is m_1, the slope of a perpendicular line is $-\dfrac{1}{m_1}$ and so

$$y - y_1 = -\frac{1}{m_1}(x - x_1)$$

is the equation of the normal at (x_1, y_1).

Example 1. Find the equations of the tangent and normal to the ellipse $x^2 + 2y^2 = 9$ at the point (1, 2).

The slope at any point of the curve is

$$\frac{dy}{dx} = -\frac{x}{2y}.$$

At (1, 2) the slope is then

$$m_1 = -\tfrac{1}{4}.$$

The equation of the tangent is

$$y - 2 = -\tfrac{1}{4}(x - 1),$$

and the equation of the normal is

$$y - 2 = 4(x - 1).$$

Ex. 2. Find the equation of the tangent to $x^2 - y^2 = a^2$ at the point (x_1, y_1).

The slope at (x_1, y_1) is $\frac{x_1}{y_1}$. The equation of the tangent is then

$$y - y_1 = \frac{x_1}{y_1}(x - x_1)$$

which reduces to

$$x_1 x - y_1 y = x_1^2 - y_1^2.$$

Since (x_1, y_1) is on the curve, $x_1^2 - y_1^2 = a^2$. The equation of the tangent can therefore be reduced to the form

$$x_1 x - y_1 y = a^2.$$

50. Angle between Two Curves. — By the angle between two curves at a point of intersection we mean the angle between their tangents at that point.

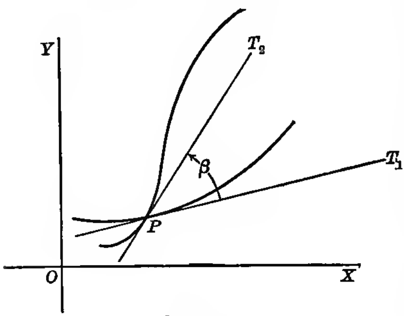

FIG. 50a.

Let m_1 and m_2 be the slopes of two curves at a point of intersection. It is shown in analytic geometry that the angle β from a line with slope m_1 to one with slope m_2 satisfies the equation

$$\tan \beta = \frac{m_2 - m_1}{1 + m_1 m_2}. \tag{50}$$

This equation thus gives the angle β from a curve with slope m_1 to one with slope m_2, the angle being considered positive when measured in the counter-clockwise direction.

Example. Find the angles determined by the line $y = x$ and the parabola $y = x^2$.

Solving the equations simultaneously, we find that the line and parabola intersect at (1, 1) and (0, 0). The slope of the line is 1. The slope at any point of the parabola is

$$\frac{dy}{dx} = 2x.$$

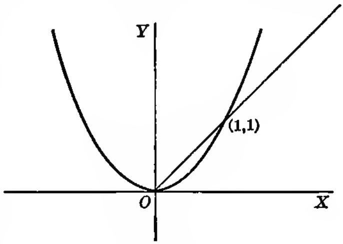

FIG. 50b.

At (1, 1) the slope of the parabola is then 2 and the angle from the line to the parabola is then given by

$$\tan \beta_1 = \frac{2-1}{1+2} = \frac{1}{3},$$

whence

$$\beta_1 = \tan^{-1}\tfrac{1}{3} = 18^\circ\, 26'.$$

At (0, 0) the slope of the parabola is 0 and so the angle from the line to the parabola is given by the equation

$$\tan \beta_2 = \frac{0-1}{1+0} = -1,$$

whence

$$\beta_2 = -45^\circ.$$

The negative sign signifies that the angle is measured in the clockwise direction from the line to the parabola.

EXERCISES

Find the tangent and normal to each of the following curves at the point indicated:

1. The circle $x^2 + y^2 = 5$ at $(-1, 2)$.
2. The hyperbola $xy = 4$ at $(1, 4)$.
3. The parabola $y^2 = ax$ at $x = a$.
4. The exponential curve $y = ab^x$ at $x = 0$.
5. The sine curve $y = 3 \sin x$ at $x = \dfrac{\pi}{6}$.
6. The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, at (x_1, y_1).

7. The hyperbola $x^2 + xy - y^2 = 2x$, at $(2, 0)$.

8. The semicubical parabola $y^3 = x^2$, at $(-8, 4)$.

9. Find the equation of the normal to the cycloid

$$x = a(\phi - \sin\phi), \qquad y = a(1 - \cos\phi)$$

at the point $\phi = \phi_1$. Show that it passes through the point where the rolling circle touches the x-axis.

Find the angles at which the following pairs of curves intersect:

10. $y^2 = 4x, \quad x^2 = 4y.$

11. $x^2 + y^2 = 9, x^2 + y^2 - 6x = 9.$

12. $x^2 + y^2 + 2x = 7, \quad y^2 = 4x.$

13. $y = \sin x, \quad y = \cos x.$

14. $y = \log_{10} x, \quad y = \ln x.$

15. $y = \frac{1}{2}(e^x + e^{-x}), \quad y = 2e^x.$

16. Show that for all values of the constants a and b the curves

$$x^2 - y^2 = a^2, \qquad xy = b^2$$

intersect at right angles.

17. Show that the curves

$$y = e^{ax}, \qquad y = e^{ax}\sin(bx + c)$$

are tangent at each point of intersection.

18. Show that the part of the tangent to the hyperbola $xy = a^2$ intercepted between the coördinate axes is bisected at the point of tangency.

19. Let the normal to the parabola $y^2 = ax$ at P cut the x-axis at N. Show that the projection of PN on the x-axis has a constant length.

20. The focus F of the parabola $y^2 = ax$ is the point $(\frac{1}{4}a, 0)$. Show that the tangent at any point P of the parabola makes equal angles with FP and the line through P parallel to the axis.

21. The foci of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad a > b$$

are the points $F'\left(-\sqrt{a^2 - b^2}, 0\right)$ and $F\left(\sqrt{a^2 - b^2}, 0\right)$. Show that the tangent at any point P of the ellipse makes equal angles with FP and $F'P$.

22. Let P be any point on the catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$, M the projection of P on the x-axis, and N the projection of M on the tangent at P. Show that MN is constant in length.

23. Show that the portion of the tangent to the tractrix

$$y = \frac{a}{2}\ln\left(\frac{a + \sqrt{a^2 - x^2}}{a - \sqrt{a^2 - x^2}}\right) - \sqrt{a^2 - x^2}$$

intercepted between the y-axis and the point of tangency is constant in length.

24. Show that the angle between the tangent at any point P and the line joining P to the origin is the same at all points of the curve

$$\ln \sqrt{x^2 + y^2} = k \tan^{-1} \frac{y}{x}.$$

25. A point at a constant distance along the normal from a given curve generates a curve which is called parallel to the first. Find the parametric equations of the parallel curve generated by the point at distance h along the normal drawn inside of the ellipse

$$x = a \cos \phi, \qquad y = b \sin \phi.$$

51. Direction of Curvature. — A curve is said to be *concave upward* at a point P if the part of the curve near P lies above the tangent at P. It is *concave downward* at Q if the part near Q lies below the tangent at Q.

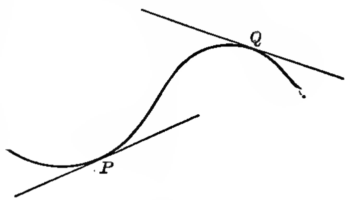

FIG. 51.

At points where $\dfrac{d^2y}{dx^2}$ *is positive, the curve is concave upward; where* $\dfrac{d^2y}{dx^2}$ *is negative, the curve is concave downward.*

For

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right).$$

If then $\dfrac{d^2y}{dx^2}$ is positive, by Art. 13, $\dfrac{dy}{dx}$, the slope, increases as x increases and decreases as x decreases. The curve therefore rises above the tangent on both sides of the point. If, however, $\dfrac{d^2y}{dx^2}$ is negative, the slope decreases as x increases and increases as x decreases, and so the curve falls below the tangent.

52. Point of Inflection. — A point like A (Fig. 52a), on one side of which the curve is concave upward, on the other concave downward, is called a *point of inflection.* It is assumed that there is a definite tangent at the point of inflection. A point like B is not called a point of inflection.

The second derivative is positive on one side of a point of inflection and negative on the other. Ordinary functions change sign only by passing through zero or infinity. Hence to find points of inflection we find where $\frac{d^2y}{dx^2}$ is zero or infinite.

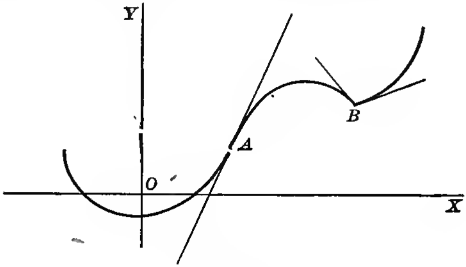

FIG. 52a.

If the second derivative changes sign at such a point, it is a point of inflection. If the second derivative has the same sign on both sides, it is not a point of inflection.

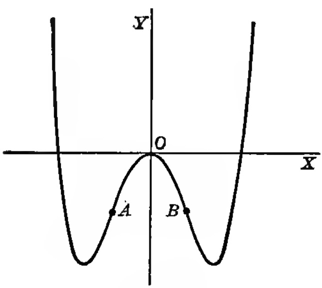

FIG. 52b.

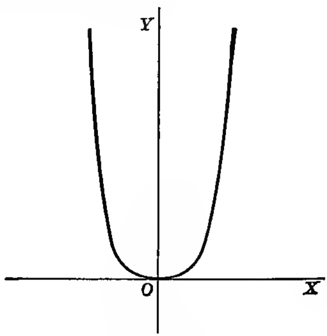

FIG. 52c.

Example 1. Examine the curve $3y = x^4 - 6x^2$ for direction of curvature and points of inflection.

The second derivative is

$$\frac{d^2y}{dx^2} = 4(x^2 - 1).$$

This is zero at $x = \pm 1$. It is positive and the curve concave upward on the left of $x = -1$ and on the right of $x = +1$. It is negative and the curve concave downward between $x = -1$ and $x = +1$. The second derivative changes sign at $A\,(-1, -\frac{5}{3})$ and $B\,(+1, -\frac{5}{3})$, which are therefore points of inflection (Fig. 52b).

Ex. 2. Examine the curve $y = x^4$ for points of inflection. In this case the second derivative is

$$\frac{d^2y}{dx^2} = 12\,x^2.$$

This is zero when x is zero but is positive for all other values of x. The second derivative does not change sign and there is consequently no point of inflection (Fig. 52c).

Ex. 3. If $x > 0$, show that $\sin x > x - \dfrac{x^3}{3!}$.*

Let

$$y = \sin x - x + \frac{x^3}{3!}.$$

We are to show that $y > 0$. Differentiation gives

$$\frac{dy}{dx} = \cos x - 1 + \frac{x^2}{2!}, \quad \frac{d^2y}{dx^2} = -\sin x + x.$$

When x is positive, $\sin x$ is less than x and so $\dfrac{d^2y}{dx^2}$ is positive. Therefore $\dfrac{dy}{dx}$ increases with x. Since $\dfrac{dy}{dx}$ is zero when x is zero, $\dfrac{dy}{dx}$ is then positive when $x > 0$, and so y increases with x. Since $y = 0$ when $x = 0$, y is therefore positive when $x > 0$, which was to be proved.

* If n is any positive integer $n!$ represents the product of the integers from 1 to n. Thus

$$3! = 1 \cdot 2 \cdot 3 = 6.$$

EXERCISES

Examine the following curves for direction of curvature and points of inflection:

1. $y = x^3 - 3x + 3.$
2. $y = 2x^3 - 3x^2 - 6x + 1.$
3. $y = x^4 - 4x^3 + 6x^2 + 12x.$
4. $y^3 = x - 1.$
5. $y = xe^x.$
6. $y = e^{-x^2}.$
7. $x^2y - 4x + 3y = 0.$
8. $x = \sin t, \quad y = \frac{1}{3}\sin 3t.$

Prove the following inequalities:

9. $x \ln x - x - \dfrac{x^2}{2} + \dfrac{3}{2} > 0, \quad$ if $0 < x < 1.$

10. $(x - 1)e^x + 1 > 0, \quad$ if $x > 0.$

11. $e^x < 1 + x + \dfrac{x^2}{2}e^a, \quad$ if $0 < x < a.$

12. $\ln \sec x > \dfrac{x^2}{2}, \quad$ if $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}.$

13. According to Van der Waal's equation, the pressure p and volume v of a gas at constant temperature T are connected by the equation

$$p = \frac{RT}{m(v - b)} - \frac{a}{v^2},$$

a, b, m, and R being constants. If p is taken as ordinate and v as abscissa, the curve represented by this equation has a point of inflection. The value of T for which the tangent at the point of inflection is horizontal is called the critical temperature. Show that the critical temperature is

$$T = \frac{8\,am}{27\,Rb}.$$

53. Length of a Curve. — The length of an arc PQ of a curve is defined as the limit (if there is a limit) approached by the length of a broken line with vertices on PQ as the number of its sides increases indefinitely, their lengths approaching zero.

We shall now show that if the slope of a curve is continuous the ratio of a chord to the arc it subtends approaches 1 as the chord approaches zero.

In the arc PQ (Fig. 53) inscribe a broken line $PABQ$. Projecting on PQ, we get

$$PQ = \text{proj. } PA + \text{proj. } AB + \text{proj. } BQ.$$

The projection of a chord, such as AB, is equal to the product of its length by the cosine of the angle it makes with PQ. On the arc AB is a tangent RS parallel to AB. Let α be the largest angle that any tangent on the arc PQ makes with the

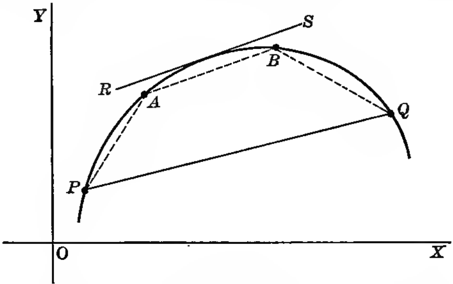

FIG. 53.

chord PQ. The angle between RS and PQ is not greater than α. Consequently, the angle between AB and PQ is not greater than α. Therefore

$$\text{proj. } AB \geqq AB \cos \alpha.$$

Similarly,

$$\text{proj. } PA \geqq PA \cos \alpha,$$
$$\text{proj. } BQ \geqq BQ \cos \alpha.$$

Adding these equations, we get

$$PQ \geqq (PA + AB + BQ) \cos \alpha.$$

It is evident that this result can be extended to a broken line with any number of sides. As the number of sides increases indefinitely, the expression in parenthesis approaches the length of the arc PQ. Therefore

$$PQ \geqq \text{arc } PQ \cos \alpha,$$

that is,

$$\frac{\text{chord } PQ}{\text{arc } PQ} \geqq \cos \alpha.$$

If the slope of the curve is continuous, the angle α approaches zero as Q approaches P. Hence $\cos \alpha$ approaches 1 and

$$\lim_{Q \doteq P} \frac{\text{chord } PQ}{\text{arc } PQ} \geqq 1.$$

Since the chord is always less than the arc, the limit cannot be greater than 1. Therefore, finally,

$$\lim_{Q \doteq P} \frac{\text{chord } PQ}{\text{arc } PQ} = 1. \tag{53}$$

54. Differential of Arc. — Let s be the distance measured along a curve from a fixed point A to a variable point P. Then s is a function of the coördinates of P. Let ϕ be the angle from the positive direction of the x-axis to the tangent PT drawn in the direction of increasing s.

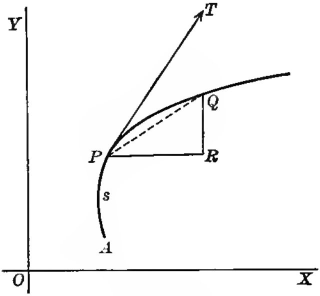

FIG. 54a.

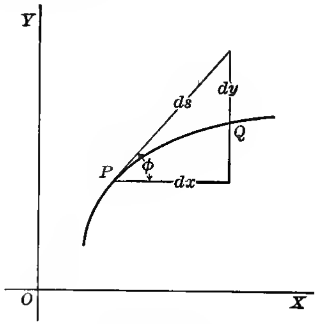

FIG. 54b.

If P moves to a neighboring position Q, the increments in x, y, and s are

$$\Delta x = PR, \qquad \Delta y = RQ, \qquad \Delta s = \text{arc } PQ.$$

From the figure it is seen that

$$\cos(RPQ) = \frac{\Delta x}{PQ} = \frac{\Delta x}{\Delta s}\frac{\Delta s}{PQ},$$

$$\sin(RPQ) = \frac{\Delta y}{PQ} = \frac{\Delta y}{\Delta s}\frac{\Delta s}{PQ}.$$

As Q approaches P, RPQ approaches ϕ and

$$\frac{\Delta s}{PQ} = \frac{\text{arc } PQ}{\text{chord } PQ}$$

approaches 1. The above equations then give in the limit

$$\cos\phi = \frac{dx}{ds}, \qquad \sin\phi = \frac{dy}{ds}. \tag{54a}$$

These equations express that dx and dy are the sides of a right triangle with hypotenuse ds extending along the tangent (Fig. 54b). All the equations connecting dx, dy, ds, and ϕ can be read off this triangle. One of particular importance is

$$ds^2 = dx^2 + dy^2. \tag{54b}$$

55. Curvature. — If an arc is everywhere concave toward its chord, the amount it is bent can be measured by the angle β between the tangents at its ends. The ratio

$$\frac{\beta}{\text{arc } PP'} = \frac{\phi' - \phi}{\Delta s} = \frac{\Delta\phi}{\Delta s}$$

is the average bending per unit length along PP'. The limit as P' approaches P,

$$\lim_{\Delta s \doteq 0} \frac{\Delta\phi}{\Delta s} = \frac{d\phi}{ds},$$

is called the *curvature* at P. It is greater where the curve bends more sharply, less where it is more nearly straight.

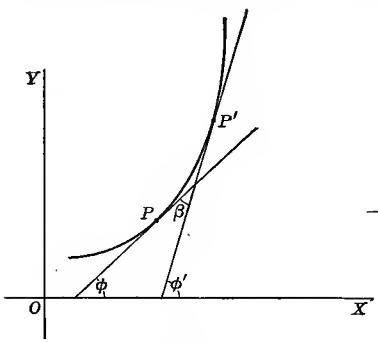

FIG. 55a.

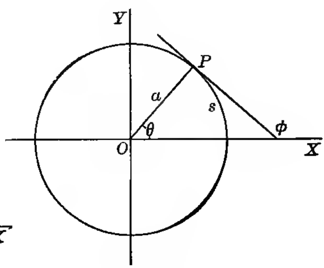

FIG. 55b.

In case of a circle (Fig. 55b)

$$\phi = \theta + \frac{\pi}{2}, \qquad s = a\theta.$$

Consequently,

$$\frac{d\phi}{ds} = \frac{d\theta}{a d\theta} = \frac{1}{a},$$

that is, *the curvature of a circle is constant and equal to the reciprocal of its radius.*

56. Radius of Curvature. — We have just seen that the radius of a circle is the reciprocal of its curvature. The *radius of curvature* of any curve is defined as the reciprocal of its curvature, that is,

$$\text{radius of curvature} = \rho = \frac{ds}{d\phi}. \tag{56a}$$

It is the radius of the circle which has the same curvature as the given curve at the given point.

To express ρ in terms of x and y we note that

$$\phi = \tan^{-1}\frac{dy}{dx}.$$

Consequently,

$$d\phi = \frac{1}{1+\left(\dfrac{dy}{dx}\right)^2}\, d\left(\frac{dy}{dx}\right) = \frac{\dfrac{d^2y}{dx^2}\,dx}{1+\left(\dfrac{dy}{dx}\right)^2}.$$

Also $$ds = \sqrt{dx^2 + dy^2}.$$

Substituting these values for ds and $d\phi$, we get

$$\rho = \frac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}. \tag{56b}$$

If the radical in the numerator is taken positive, ρ will have the same sign as $\dfrac{d^2y}{dx^2}$, that is, the radius will be positive when the curve is concave upward. If merely the numerical value is wanted, the sign can be omitted.

By a similar proof we could show that

$$\rho = \frac{\left[1+\left(\dfrac{dx}{dy}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2x}{dy^2}}. \tag{56c}$$

Example 1. Find the radius of curvature of the parabola $y^2 = 4x$ at the point (4, 4).

At the point (4, 4) the derivatives have the values

$$\frac{dy}{dx} = \frac{2}{y} = \frac{1}{2}, \qquad \frac{d^2y}{dx^2} = -\frac{4}{y^3} = -\frac{1}{16}.$$

Therefore

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left(1 + \frac{1}{4}\right)^{\frac{3}{2}}}{-\frac{1}{16}} = -10\sqrt{5}.$$

The negative sign shows that the curve is concave downward. The length of the radius is $10\sqrt{5}$.

Ex. 2. Find the radius of curvature of the curve represented by the polar equation $r = a \cos\theta$.

The expressions for x and y in terms of θ are

$$x = r\cos\theta = a\cos\theta\cos\theta = a\cos^2\theta,$$
$$y = r\sin\theta = a\cos\theta\sin\theta.$$

Consequently,

$$\frac{dy}{dx} = \frac{a(\cos^2\theta - \sin^2\theta)}{-2a\cos\theta\sin\theta} = \frac{a\cos 2\theta}{-a\sin 2\theta} = -\cot 2\theta,$$

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = -\frac{2\csc^2 2\theta\, d\theta}{a\sin 2\theta\, d\theta} = -\frac{2}{a}\csc^3 2\theta.$$

$$\rho = \frac{[1 + \cot^2 2\theta]^{\frac{3}{2}}}{-\frac{2}{a}\csc^3 2\theta} = -a\frac{(\csc^2 2\theta)^{\frac{3}{2}}}{2\csc^3 2\theta} = -\frac{a}{2}.$$

The radius is thus constant. The curve is in fact a circle.

57. Center and Circle of Curvature. — At each point of a curve is a circle on the concave side tangent at the point with radius equal to the radius of curvature. This circle is called the *circle of curvature*. Its center is called the *center of curvature*.

Since the circle and curve are tangent at P, they have the same slope $\frac{dy}{dx}$ at P. Since they have the same radius of curvature, the second derivatives will also be equal at P.

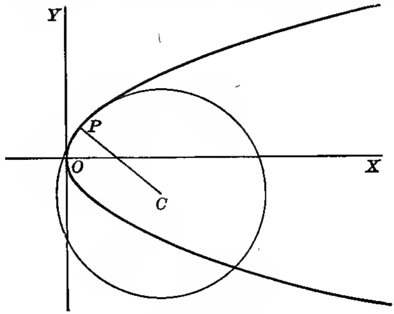

FIG. 57.

The circle of curvature is thus the circle through P such that $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ have the same values for the circle as for the curve at P.

EXERCISES

1. The length of arc measured from a fixed point on a certain curve is $s = x^2 + x$. Find the slope of the curve at $x = 2$.

2. Can $x = \cos s$, $y = \sin s$, represent a curve on which s is the length of arc measured from a fixed point? Can $x = \sec s$, $y = \tan s$, represent such a curve?

Find the radius of curvature on each of the following curves at the point indicated:

3. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, at $(0, b)$.

4. $x^2 + xy + y^2 = 3$, at $(1, 1)$.

5. $r = e^\theta$, at $\theta = \frac{\pi}{2}$.

6. $r = a(1 + \cos\theta)$, at $\theta = 0$.

Find an expression for the radius of curvature at any point of each of the following curves:

7. $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$.

8. $x = \ln \sec y$.

9. $x = \frac{1}{4} y^2 - \frac{1}{2} \ln y$.

10. $r = a \sec^2 \frac{1}{2}\theta$.

11. Show that the radius of curvature at a point of inflection is infinite.

12. A point on the circumference of a circle rolling along the x-axis generates the cycloid

$$x = a(\phi - \sin\phi), \qquad y = a(1 - \cos\phi),$$

a being the radius of the rolling circle and ϕ the angle through which it has turned. Show that the radius of the circle of curvature is bisected by the point where the rolling circle touches the x-axis.

13. A string held taut is unwound from a fixed circle. The end of the string generates a curve with parametric equations

$$x = a\cos\theta + a\theta\sin\theta, \qquad y = a\sin\theta - a\theta\cos\theta,$$

a being the radius of the circle and θ the angle subtended at the center by the arc unwound. Show that the center of curvature corresponding to any point of this path is the point where the string is tangent to the circle.

14. Show that the radius of curvature at any point (x, y) of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is three times the perpendicular from the origin to the tangent at (x, y).

58. Limit of $\dfrac{1 - \cos x}{x}$. It is shown in trigonometry that

$$1 - \cos x = 2\sin^2\frac{x}{2}.$$

Consequently,

$$\frac{1 - \cos x}{x} = \frac{2\sin^2\frac{x}{2}}{x} = \sin\frac{x}{2}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right).$$

As x approaches zero, $\dfrac{\sin\frac{x}{2}}{\frac{x}{2}}$ approaches 1. Therefore

$$\lim_{x \doteq 0}\frac{1 - \cos x}{x} = 0 \cdot 1 = 0.$$

59. Derivatives of Arc in Polar Coördinates. — The angle from the outward drawn radius to the tangent drawn in the direction of increasing s is usually represented by the letter ψ.

Let r, θ be the polar coördinates of P, and $r + \Delta r$, $\theta + \Delta\theta$ those of Q (Fig. 59a). Draw QR perpendicular to PR and let Δs = arc PQ. Then

$$\sin(RPQ) = \frac{RQ}{PQ} = \frac{(r+\Delta r)\sin\Delta\theta}{PQ} = (r+\Delta r)\frac{\sin\Delta\theta}{\Delta\theta}\cdot\frac{\Delta\theta}{\Delta s}\cdot\frac{\Delta s}{PQ}.$$

$$\begin{aligned}\cos(RPQ) &= \frac{PR}{PQ} = \frac{(r+\Delta r)\cos\Delta\theta - r}{PQ}\\ &= \cos(\Delta\theta)\frac{\Delta r}{PQ} - \frac{r(1-\cos\Delta\theta)}{PQ}\\ &= \cos(\Delta\theta)\frac{\Delta r}{\Delta s}\cdot\frac{\Delta s}{PQ} - \frac{r(1-\cos\Delta\theta)}{\Delta\theta}\frac{\Delta\theta}{\Delta s}\cdot\frac{\Delta s}{PQ}.\end{aligned}$$

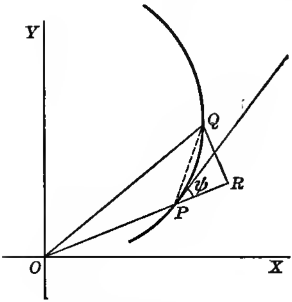

FIG. 59a.

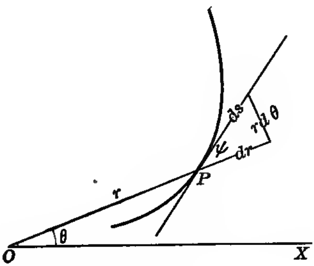

FIG. 59b.

As $\Delta\theta$ approaches zero,

$$\lim(RPQ) = \psi,\quad \lim\frac{\sin\Delta\theta}{\Delta\theta} = 1,\quad \lim\frac{1-\cos\Delta\theta}{\Delta\theta} = 0,\quad \lim\frac{\Delta s}{PQ} = 1.$$

The above equations then give in the limit,

$$\sin\psi = \frac{r d\theta}{ds},\qquad \cos\psi = \frac{dr}{ds}. \tag{59a}$$

These equations show that dr and $r\,d\theta$ are the sides of a right triangle with hypotenuse ds and base angle ψ. From this triangle all the equations connecting dr, $d\theta$, ds, and ψ can be obtained. The most important of these are

$$\tan\psi = \frac{r\,d\theta}{dr},\qquad ds^2 = dr^2 + r^2\,d\theta^2. \tag{59b}$$

Example. The logarithmic spiral $r = ae^{\theta}$.

In this case, $dr = ae^{\theta}\,d\theta$ and so

$$\tan \psi = \frac{r\,d\theta}{dr} = 1.$$

The angle ψ is therefore constant and equal to 45 degrees. The equation

$$\cos \psi = \frac{dr}{ds} = \frac{1}{\sqrt{2}}$$

shows that $\frac{dr}{ds}$ is also constant and so r and s increase proportionally.

EXERCISES

Find the angle ψ at the point indicated on each of the following curves:

1. The spiral $r = a\theta$, at $\theta = \frac{\pi}{3}$.

2. The circle $r = a \sin \theta$ at $\theta = \frac{\pi}{4}$.

3. The straight line $r = a \sec \theta$, at $\theta = \frac{\pi}{6}$.

4. The ellipse $r\,(2 - \cos \theta) = k$, at $\theta = \frac{\pi}{2}$.

5. The lemniscate $r^2 = 2\,a^2 \cos 2\,\theta$, at $\theta = \frac{5}{6}\,\pi$.

6. Show that the curves $r = ae^{\theta}$, $r = ae^{-\theta}$ are perpendicular at each of their points of intersection.

7. Find the angles at which the curves $r = a \cos \theta$, $r = a \sin 2\,\theta$ intersect.

8. Find the points on the cardioid $r = a\,(1 - \cos \theta)$ where the tangent is parallel to the initial line.

9. Let $P\,(r, \theta)$ be a point on the hyperbola $r^2 \sin 2\,\theta = c$. Show that the triangle formed by the radius OP, the tangent at P, and the x-axis is isosceles.

10. Find the slope of the curve $r = e^{2\theta}$ at the point where $\theta = \frac{\pi}{4}$.

60. Angle between Two Directed Lines in Space. — A directed line is one along which a positive direction is assigned. This direction is usually indicated by an arrow.

An angle between two directed lines is one along the sides of which the arrows point away from the vertex. There are two such angles less than 360 degrees, their sum being 360 degrees (Fig. 60). They have the same cosine.

If the lines do not intersect, the angle between them is defined as that between intersecting lines respectively parallel to the given lines.

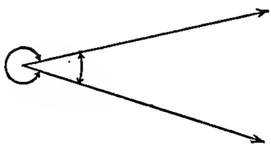

FIG. 60.

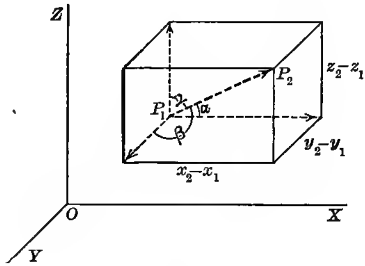

FIG. 61.

61. Direction Cosines. — It is shown in analytic geometry* that the angles α, β, γ between the coördinate axes and the line P_1P_2 (directed from P_1 to P_2) satisfy the equations

$$\cos\alpha = \frac{x_2 - x_1}{P_1P_2}, \quad \cos\beta = \frac{y_2 - y_1}{P_1P_2}, \quad \cos\gamma = \frac{z_2 - z_1}{P_1P_2}. \qquad (61a)$$

These cosines are called the *direction cosines* of the line. They satisfy the identity

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \qquad (61b)$$

If the direction cosines of two lines are $\cos\alpha_1$, $\cos\beta_1$, $\cos\gamma_1$ and $\cos\alpha_2$, $\cos\beta_2$, $\cos\gamma_2$, the angle θ between the lines is given by the equation

$$\cos\theta = \cos\alpha_1 \cos\alpha_2 + \cos\beta_2 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2. \qquad (61c)$$

In particular, if the lines are perpendicular, the angle θ is 90 degrees and

$$0 = \cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2. \qquad (61d)$$

* Cf. H. B. Phillips, *Analytic Geometry*, Art. 64, et seq.

62. Direction of the Tangent Line to a Curve. — The tangent line at a point P of a curve is defined as the limiting position PT approached by the secant PQ as Q approaches P along the curve.

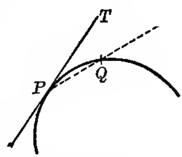

FIG. 62a.

Let s be the arc of the curve measured from some fixed point and $\cos\alpha$, $\cos\beta$, $\cos\gamma$ the direction cosines of the tangent drawn in the direction of increasing s.

If x, y, z are the coördinates of P, $x + \Delta x$, $y + \Delta y$, $z + \Delta z$, those of Q, the direction cosines of PQ are

$$\frac{\Delta x}{PQ},\quad \frac{\Delta y}{PQ},\quad \frac{\Delta z}{PQ}.$$

As Q approaches P, these approach the direction cosines of the tangent at P. Hence

$$\cos\alpha = \lim_{Q \doteq P} \frac{\Delta x}{PQ} = \lim \frac{\Delta x}{\Delta s}\,\frac{\Delta s}{PQ}.$$

On the curve, x, y, z are functions of s. Hence

$$\lim \frac{\Delta x}{\Delta s} = \frac{dx}{ds};\qquad \lim \frac{\Delta s}{PQ} = \lim \frac{\text{arc}}{\text{chord}} = 1.*$$

Therefore

$$\cos\alpha = \frac{dx}{ds}. \tag{62a}$$

Similarly,

$$\cos\beta = \frac{dy}{ds},\qquad \cos\gamma = \frac{dz}{ds}. \tag{62a}$$

These equations show that if a distance ds is measured along the tangent, dx, dy, dz are its projections on the coördinate axes (Fig. 62b). Since the square on the diagonal of a

* The proof that the limit of arc/chord is 1 was given in Art. 53 for the case of plane curves with continuous slope. A similar proof can be given for any curve, plane or space, that is continuous in direction.

rectangular parallelopiped is equal to the sum of the squares of its three edges,

$$ds^2 = dx^2 + dy^2 + dz^2. \tag{62b}$$

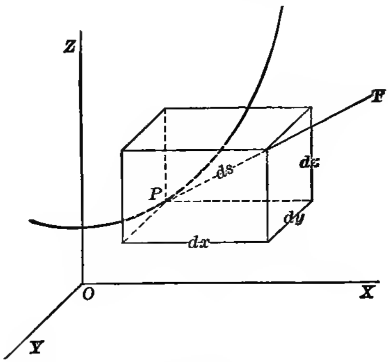

FIG. 62b.

Example. Find the direction cosines of the tangent to the parabola

$$x = at, \qquad y = bt, \qquad z = \tfrac{1}{4}\, ct^2$$

at the point where $t = 2$.

At $t = 2$ the differentials are

$$dx = a\, dt, \qquad dy = b\, dt, \qquad dz = \tfrac{1}{2}\, ct\, dt = c\, dt,$$

$$ds = \pm\sqrt{dx^2 + dy^2 + dz^2} = \pm\sqrt{a^2 + b^2 + c^2}\, dt.$$

There are two algebraic signs depending on the direction s is measured along the curve. If we take the positive sign, the direction cosines are

$$\frac{dx}{ds} = \frac{a}{\sqrt{a^2 + b^2 + c^2}}, \qquad \frac{dy}{ds} = \frac{b}{\sqrt{a^2 + b^2 + c^2}},$$

$$\frac{dz}{ds} = \frac{c}{\sqrt{a^2 + b^2 + c^2}}.$$

63. Equations of the Tangent Line. — It is shown in analytic geometry that the equations of a straight line through a point P_1 (x_1, y_1, z_1) with direction cosines proportional to A, B, C are

$$\frac{x - x_1}{A} = \frac{y - y_1}{B} = \frac{z - z_1}{C}. \tag{63}$$

The direction cosines of the tangent line are proportional to dx, dy, dz. If then we replace A, B, C by numbers proportional to the values of dx, dy, dz at P_1, (63) will represent the tangent line at P_1.

Example 1. Find the equations of the tangent to the curve

$$x = t, \qquad y = t^2, \qquad z = t^3$$

at the point where $t = 1$.

The point of tangency is $t = 1$, $x_1 = 1$, $y_1 = 1$, $z_1 = 1$. At this point the differentials are

$$dx : dy : dz = dt : 2\,t\,dt : 3\,t^2\,dt = 1 : 2 : 3.$$

The equations of the tangent line are then

$$\frac{x - 1}{1} = \frac{y - 1}{2} = \frac{z - 1}{3}.$$

Ex. 2. Find the angle between the curve $3\,x + 2\,y - 2\,z = 3$, $4\,x^2 + y^2 = 2\,z^2$ and the line joining the origin to $(1, 2, 2)$.

The curve and line intersect at $(1, 2, 2)$. Along the curve y and z can be considered functions of x. The differentials satisfy the equations

$$3\,dx + 2\,dy - 2\,dz = 0, \quad 8\,x\,dx + 2\,y\,dy = 4\,z\,dz.$$

At the point of intersection these equations become

$$3\,dx + 2\,dy - 2\,dz = 0, \quad 8\,dx + 4\,dy = 8\,dz.$$

Solving for dx and dy in terms of dz, we get

$$dx = 2\,dz, \qquad dy = -2\,dz.$$

Consequently,

$$ds = \sqrt{dx^2 + dy^2 + dz^2} = 3\,dz$$

and

$$\cos\alpha = \frac{dx}{ds} = \frac{2}{3}, \qquad \cos\beta = \frac{dy}{ds} = \frac{-2}{3}, \qquad \cos\gamma = \frac{dz}{ds} = \frac{1}{3}.$$

The line joining the origin and (1, 2, 2) has direction cosines equal to

$$\tfrac{1}{3},\ \tfrac{2}{3},\ \tfrac{2}{3}.$$

The angle θ between the line and curve satisfies the equation

$$\cos\theta = \frac{2 - 4 + 2}{9} = 0.$$

The line and curve intersect at right angles.

EXERCISES

Find the equations of the tangent lines to the following curves at the points indicated:

1. $x = \sec t, \quad y = \tan t, \quad z = at, \quad \text{at } t = \frac{\pi}{4}.$

2. $x = e^t, \quad y = e^{-t}, \quad z = t^2, \quad \text{at } t = 1.$

3. $x = e^t \sin t, \quad y = e^t \cos t, \quad z = kt, \quad \text{at } t = \frac{\pi}{2}.$

4. On the circle

$$x = a\cos\theta, \quad y = a\cos\left(\theta + \frac{2}{3}\pi\right), \quad z = a\cos\left(\theta + \frac{4}{3}\pi\right)$$

show that ds is proportional to $d\theta$.

5. Find the angle at which the helix

$$x = a\cos\theta, \quad y = a\sin\theta, \quad z = k\theta$$

cuts the generators of the cylinder $x^2 + y^2 = a^2$ on which it lies.

6. Find the angle at which the conical helix

$$x = t\cos t, \quad y = t\sin t, \quad z = t$$

cuts the generators of the cone $x^2 + y^2 = z^2$ on which it lies.

7. Find the angle between the two circles cut from the sphere $x^2 + y^2 + z^2 = 14$ by the planes $x - y + z = 0$ and $x + y - z = 2$.

CHAPTER VIII

VELOCITY AND ACCELERATION IN A CURVED PATH

64. Speed of a Particle. — When a particle moves along a curve, its speed is the rate of change of distance along the path.

Let a particle P move along the curve AB, Fig. 64. Let s be the arc from a fixed point A to P. The speed of the particle is then

$$v = \frac{ds}{dt}. \tag{64}$$

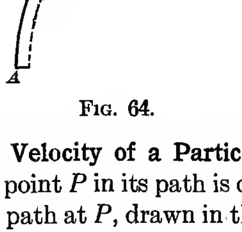

FIG. 64.

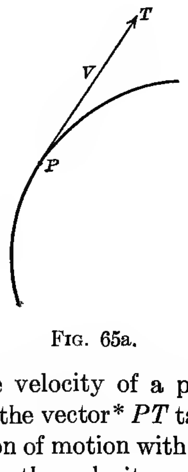

FIG. 65a.

65. Velocity of a Particle. — The velocity of a particle at the point P in its path is defined as the vector* PT tangent to the path at P, drawn in the direction of motion with length equal to the speed at P. To specify the velocity we must then give the speed and direction of motion.

* A vector is a quantity having length and direction. The direction is usually indicated by an arrow. Two vectors are called equal when they extend along the same line or along parallel lines and have the same length and direction.

The particle can be considered as moving instantaneously in the direction of the tangent. The velocity indicates in magnitude and direction the distance it would move in a unit of time if the speed and direction of motion did not change.

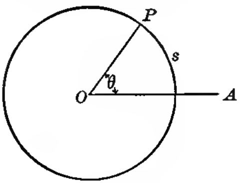

FIG. 65b.

Example. A wheel 4 ft. in diameter rotates at the rate of 500 revolutions per minute. Find the speed and velocity of a point on its rim.

Let OA be a fixed line through the center of the wheel and s the distance along the wheel from OA to a moving point P. Then

$$s = 2\,\theta \text{ ft.}$$

The speed of P is

$$\frac{ds}{dt} = 2\frac{d\theta}{dt} = 2\,(500)\,2\,\pi = 2000\,\pi \text{ ft./min.}$$

Its velocity is $2000\,\pi$ ft./min. *in the direction of the tangent at P.* The speeds of all points on the rim are the same. Their velocities differ in direction.

66. Components of Velocity in a Plane. — To specify a velocity in a plane it is customary to give its components, that is, its projections on the coördinate axes.

If PT is the velocity at P (Fig. 66), the x-component is

$$PQ = PT\cos\phi = \frac{ds}{dt}\cos\phi = \frac{ds}{dt}\frac{dx}{ds} = \frac{dx}{dt},$$

and the y-component is

$$QT = PT\sin\phi = \frac{ds}{dt}\sin\phi = \frac{ds}{dt}\frac{dy}{ds} = \frac{dy}{dt}$$

The components are thus the rates of change of the coördinates.

Since

$$PT^2 = PQ^2 + QT^2,$$

the speed is expressed in terms of the components by the equation

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2.$$

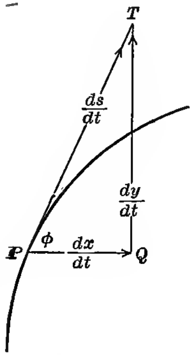

FIG. 66.

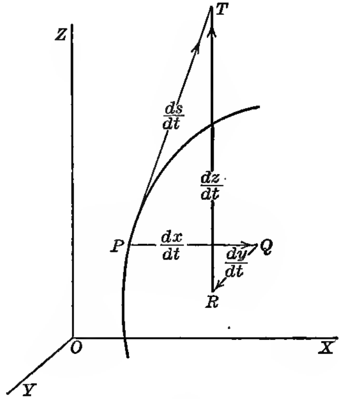

FIG. 67.

67. Components in Space. — If a particle is moving along a space curve, the projections of its velocity on the three coördinate axes are called components.

Thus, if PT (Fig. 67) represents the velocity of a point, its components are

$$PQ = PT \cos \alpha = \frac{ds}{dt}\frac{dx}{ds} = \frac{dx}{dt},$$

$$QR = PT \cos \beta = \frac{ds}{dt}\frac{dy}{ds} = \frac{dy}{dt},$$

$$RT = PT \cos \gamma = \frac{ds}{dt}\frac{dz}{ds} = \frac{dz}{dt}.$$

Since $PT^2 = PQ^2 + QR^2 + RT^2$, the speed and components are connected by the equation

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2.$$

68. Notation. — In this book we shall indicate a vector with given components by placing the components in brackets. Thus to indicate that a velocity has an x-component equal to 3 and a y-component equal to -2, we shall simply say that the velocity is $[3, -2]$. Similarly, a vector in space with x-component a, y-component b, and z-component c will be represented by the symbol $[a, b, c]$.

Example 1. Neglecting the resistance of the air a bullet fired with a velocity of 1000 ft. per second at an angle of 30 degrees with the horizontal plane will move a horizontal distance

$$x = 500\,t\,\sqrt{3}$$

and a vertical distance

$$y = 500\,t - 16.1\,t^2$$

in t seconds. Find its velocity and speed at the end of 10 seconds.

The components of velocity are

$$\frac{dx}{dt} = 500\,\sqrt{3}, \qquad \frac{dy}{dt} = 500 - 32.2\,t.$$

At the end of 10 seconds the velocity is then

$$V = [\,500\,\sqrt{3},\, 178]$$

and the speed is

$$\frac{ds}{dt} = \sqrt{(500\,\sqrt{3})^2 + (178)^2} = 884 \text{ ft./sec.}$$

Ex. 2. A point on the thread of a screw which is turned into a fixed nut describes a helix with equations

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = k\theta,$$

θ being the angle through which the screw has turned, r the radius, and k the pitch of the screw. Find the velocity and speed of the point.

The components of velocity are

$$\frac{dx}{dt} = -r \sin\theta \frac{d\theta}{dt}, \quad \frac{dy}{dt} = r\cos\theta \frac{d\theta}{dt}, \quad \frac{dz}{dt} = k\frac{d\theta}{dt}.$$

Since $\frac{d\theta}{dt}$ is the angular velocity ω with which the screw is rotating, the velocity of the moving point is

$$V = [-r\omega \sin\theta, r\omega\cos\theta, k\omega]$$

and its speed is

$$\frac{ds}{dt} = \sqrt{r^2\omega^2 \sin^2\theta + r^2\omega^2\cos^2\theta + k^2\omega^2} = \omega\sqrt{r^2 + k^2},$$

which is constant.

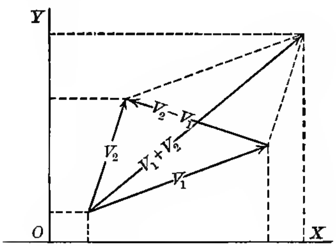

FIG. 69a.

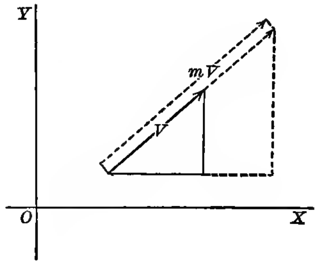

FIG. 69b.

69. Composition of Velocities. — By the sum of two velocities V_1 and V_2 is meant the velocity $V_1 + V_2$ whose components are obtained by adding corresponding components of V_1 and V_2. Similarly, the difference $V_2 - V_1$ is the velocity whose components are obtained by subtracting the components of V_1 from the corresponding ones of V_2.

Thus, if

$$V_1 = [a_1, b_1], \qquad V_2 = [a_2, b_2],$$
$$V_1 + V_2 = [a_1 + a_2, b_1 + b_2], \qquad V_2 - V_1 = [a_2 - a_1, b_2 - b_1].$$

If V_1 and V_2 extend from the same point (Fig. 69a), $V_1 + V_2$ is one diagonal of the parallelogram with V_1 and V_2 as adjacent sides and $V_2 - V_1$ is the other. In this case $V_2 - V_1$ extends from the end of V_1 to the end of V_2.

By the product mV of a vector by a number is meant a vector m times as long as V and extending in the same direction if m is positive but the opposite direction if m is negative. It is evident from Fig. 69b that the components of mV are m times those of V.

The quotient $\frac{V}{m}$ can be considered as a product $\frac{1}{m}V$. Its components are obtained by dividing those of V by m.

70. Acceleration. — The acceleration of a particle moving along a curved path is the rate of change of its velocity

$$A = \lim_{\Delta t \doteq 0} \frac{\Delta V}{\Delta t} = \frac{dV}{dt}.$$

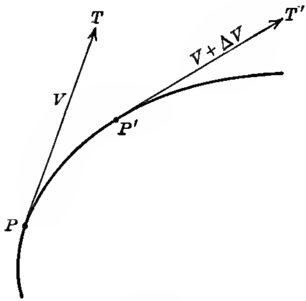

FIG. 70a.

In this equation ΔV is a vector and $\frac{\Delta V}{\Delta t}$ is obtained by dividing the components of ΔV by Δt.

Let the particle move from the point P where the velocity is V to an adjacent point P' where the velocity is $V + \Delta V$. The components of velocity will change from $\frac{dx}{dt}, \frac{dy}{dt}$ to

$$\frac{dx}{dt} + \Delta\frac{dx}{dt}, \quad \frac{dy}{dt} + \Delta\frac{dy}{dt}.$$

Consequently,

$$V = \left[\frac{dx}{dt}, \frac{dy}{dt}\right], \quad V + \Delta V = \left[\frac{dx}{dt} + \Delta\frac{dx}{dt}, \frac{dy}{dt} + \Delta\frac{dy}{dt}\right].$$

Subtraction and division by Δt give

$$\Delta V = \left[\Delta\frac{dx}{dt}, \Delta\frac{dy}{dt}\right], \quad \frac{\Delta V}{\Delta t} = \left[\frac{\Delta\frac{dx}{dt}}{\Delta t}, \frac{\Delta\frac{dy}{dt}}{\Delta t}\right].$$

As Δt approaches zero, the last equation approaches

$$A = \frac{dV}{dt} = \left[\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}\right]. \qquad (70a)$$

In the same way the acceleration of a particle moving in space is found to be

$$A = \left[\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\right]. \tag{70b}$$

Equations 70a and 70b express that *the components of the acceleration of a moving particle are the second derivatives of its coördinates with respect to the time.*

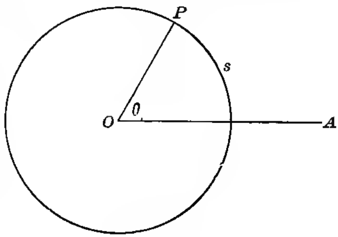

FIG. 70b.

Example. A particle moves with a constant speed v around a circle of radius r. Find its velocity and acceleration at each point of the path.

Let $\theta = AOP$. The coördinates of P are

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

The velocity of P is

$$V = \left[-r\sin\theta\frac{d\theta}{dt}, r\cos\theta\frac{d\theta}{dt}\right].$$

Since $s = r\theta$, $\dfrac{ds}{dt} = v = r\dfrac{d\theta}{dt}$. The velocity can therefore be written

$$V = [-v\sin\theta, \quad v\cos\theta].$$

Since v is constant, the acceleration is

$$A = \frac{dV}{dt} = \left[\frac{d}{dt}(-v\sin\theta), \ \frac{d}{dt}(v\cos\theta)\right]$$

$$= \left[-v\cos\theta\frac{d\theta}{dt}, \ -v\sin\theta\frac{d\theta}{dt}\right].$$

Replacing $\dfrac{d\theta}{dt}$ by $\dfrac{v}{r}$, this reduces to

$$A = \left[-\frac{v^2}{r}\cos\theta, \quad -\frac{v^2}{r}\sin\theta\right] = \frac{v^2}{r}[-\cos\theta, -\sin\theta].$$

Now $[-\cos\theta, \ -\sin\theta]$ is a vector of unit length directed along PO toward the center. Hence the acceleration of P is directed toward the center of the circle and has a magnitude equal to $\frac{v^2}{r}$.

EXERCISES

1. A point P moves with constant speed v along the straight line $y=a$. Find the speed with which the line joining P to the origin rotates.

2. A rod of length a slides with its ends in the x- and y-axes. If the end in the x-axis moves with constant speed v, find the velocity and speed of the middle point of the rod.

3. A wheel of radius a rotates about its center with angular speed ω while the center moves along the x-axis with velocity v. Find the velocity and speed of a point on the perimeter of the wheel.

4. Two particles P_1 (x_1, y_1) and P_2 (x_2, y_2) move in such a way that

$$x_1 = 1 + 2t, \qquad y_1 = 2 - 3t^2,$$
$$x_2 = 3 + 2t^2, \qquad y_2 = -4t^3.$$

Find the two velocities and show that they are always parallel.

5. Two particles P_1 (x_1, y_1, z_1) and P_2 (x_2, y_2, z_2) move in such a way that

$$x_1 = a\cos\theta,\ y_1 = a\cos(\theta + \tfrac{1}{3}\pi),\ z_1 = a\cos(\theta + \tfrac{2}{3}\pi),$$
$$x_2 = a\sin\theta,\ y_2 = a\sin(\theta + \tfrac{1}{3}\pi),\ z_2 = a\sin(\theta + \tfrac{2}{3}\pi).$$

Find the two velocities and show that they are always at right angles.

6. A man can row 3 miles per hour and walk 4. He wishes to cross a river and arrive at a point 6 miles further up the river. If the river is $1\frac{2}{3}$ miles wide and the current flows 2 miles per hour, find the course he shall take to reach his destination in the least time.

7. Neglecting the resistance of the air a projectile fired with velocity $[a, b, c]$ moves in t seconds to a position

$$x = at, \quad y = bt, \quad z = ct - \tfrac{1}{2}gt^2.$$

Find its speed, velocity, and acceleration.

8. A particle moves along the parabola $x^2 = ay$ in such a way that $\frac{dx}{dt}$ is constant. Show that its acceleration is constant.

9. When a wheel rolls along a straight line, a point on its circumference describes a cycloid with parametric equations

$$x = a(\phi - \sin\phi), \qquad y = a(1 - \cos\phi),$$

a being the radius of the wheel and ϕ the angle through which it has rotated. Find the speed, velocity, and acceleration of the moving point.

10. Find the acceleration of a particle moving with constant speed v along the cardioid $r = a(1 - \cos\theta)$.

11. If a string is held taut while it is unwound from a fixed circle, its end describes the curve

$$x = a\cos\theta + a\,\theta\sin\theta,\ y = a\sin\theta - a\,\theta\cos\theta,$$

θ being the angle subtended at the center by the arc unwound. Show that the end moves at each instant with the same velocity it would have if the straight part of the string rotated with angular velocity $\frac{d\theta}{dt}$ about the point where it meets the fixed circle.

12. A piece of mechanism consists of a rod rotating in a plane with constant angular velocity ω about one end and a ring sliding along the rod with constant speed v. (1) If when $t = 0$ the ring is at the center of rotation, find its position, velocity, and acceleration as functions of the time. (2) Find the velocity and acceleration immediately after $t = t_1$, if at that instant the rod ceases to rotate but the ring continues sliding with unchanged speed along the rod. (3) Find the velocity and acceleration immediately after $t = t_1$ if at that instant the ring ceases sliding but the rod continues rotating. (4) How are the three velocities related? How are the three accelerations related?

13. Two rods AB, BC are hinged at B and lie in a plane. A is fixed, AB rotates with angular speed ω about A, and BC rotates with angular speed 2ω about B. (1) If when $t = 0$, C lies on AB produced, find the path, velocity, and acceleration of C. (2) Find the velocities and accelerations immediately after $t = t_1$ if at that instant one of the rotations ceases. (3) How are the actual velocity and acceleration related to these partial velocities and accelerations?

14. A hoop of radius a rolls with angular velocity ω_1 along a horizontal line, while an insect crawls along the rim with speed $a\omega_2$. If when $t = 0$ the insect is at the bottom of the hoop, find its path, velocity, and acceleration. The motion of the insect results from three simultaneous actions, the advance of the center of the hoop with speed $a\omega_1$, the rotation of the hoop about its center with angular speed ω_1, and the crawl of the insect advancing its radius with angular speed ω_2. Find the three velocities and accelerations which result if at the time $t = t_1$ two of these actions cease, the third continuing unchanged. How are the actual velocity and acceleration related to these partial velocities and accelerations?

CHAPTER IX

ROLLE'S THEOREM AND INDETERMINATE FORMS

71. Rolle's Theorem. — *If $f'(x)$ is continuous, there is at least one real root of $f'(x) = 0$ between each pair of real roots of $f(x) = 0$.*

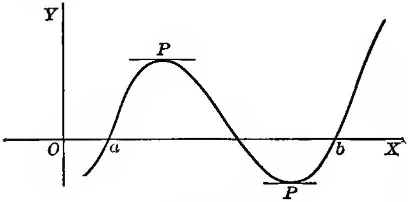

FIG. 71a.

To show this consider the curve

$$y = f(x).$$

Let $f(x)$ be zero at $x = a$ and $x = b$. Between a and b there must be one or more points P at maximum distance from the x-axis. At such a point the tangent is horizontal and so

$$\frac{dy}{dx} = f'(x) = 0.$$

That this theorem may not hold if $f'(x)$ is discontinuous is shown in Figs. 71b and 71c. In both cases the curve

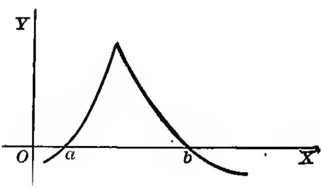

FIG. 71b.

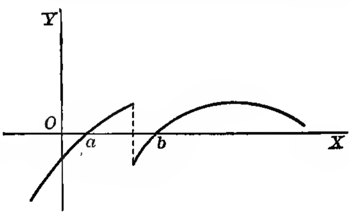

FIG. 71c.

crosses the x-axis at a and b but there is no intermediate point where the slope is zero.

Example. Show that the equation

$$x^3 + 3x - 6 = 0$$

cannot have more than one real root.

Let

$$f(x) = x^3 + 3x - 6.$$

Then

$$f'(x) = 3x^2 + 3 = 3(x^2 + 1).$$

Since $f'(x)$ does not vanish for any real value of x, $f(x) = 0$ cannot have more than one real root; for if there were two there would be a root of $f'(x) = 0$ between them.

72. Indeterminate Forms. — The expressions

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0 \cdot \infty, \quad \infty - \infty, \quad 1^\infty, \quad 0^0, \quad \infty^0$$

are called indeterminate forms. No definite values can be assigned to them.

If when $x = a$ a function $f(x)$ assumes an indeterminate form, there may however be a definite limit

$$\lim_{x \doteq a} f(x).$$

In such cases this limit is usually taken as the value of the function at $x = a$.

For example, when $x = 0$ the function

$$\frac{2x}{x} = \frac{0}{0}.$$

It is evident, however, that

$$\lim_{x \doteq 0} \frac{2x}{x} = \lim (2) = 2.$$

This example shows that an indeterminate form can often be made definite by an algebraic change of form.

73. The Forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$. — We shall now show that, if for a particular value of the variable a fraction $\dfrac{f(x)}{F(x)}$ assumes the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$, numerator and denominator can be replaced by their derivatives without changing the value of the limit approached by the fraction as x approaches a.

1. Let $f'(x)$ and $F'(x)$ be continuous between a and b. *If* $f(a) = 0$, $F(a) = 0$, *and* $F(b)$ *is not zero, there is a number* x_1 *between* a *and* b *such that*

$$\frac{f(b)}{F(b)} = \frac{f'(x_1)}{F'(x_1)}. \tag{73a}$$

To show this let $\dfrac{f(b)}{F(b)} = R$. Then

$$f(b) - RF(b) = 0.$$

Consider the function

$$f(x) - RF(x).$$

This function vanishes when $x = b$. Since $f(a) = 0$, $F(a) = 0$, it also vanishes when $x = a$. By Rolle's Theorem there is then a value x_1 between a and b such that

$$f'(x_1) - RF'(x_1) = 0.$$

Consequently,

$$\frac{f(b)}{F(b)} = R = \frac{f'(x_1)}{F'(x_1)},$$

which was to be proved.

2. Let $f'(x)$ and $F'(x)$ be continuous near a. *If* $f(a) = 0$ *and* $F(a) = 0$, *then*

$$\lim_{x \doteq a} \frac{f(x)}{F(x)} = \lim_{x \doteq a} \frac{f'(x)}{F'(x)}. \tag{73b}$$

For, if we replace b by x, (73a) becomes

$$\frac{f(x)}{F(x)} = \frac{f'(x_1)}{F'(x_1)},$$

x_1 being between a and x. Since x_1 approaches a as x approaches a,

$$\lim_{x \doteq a} \frac{f(x)}{F(x)} = \lim_{x_1 \doteq a} \frac{f'(x_1)}{F'(x_1)} = \lim_{x \doteq a} \frac{f'(x)}{F'(x)}.$$

3. In the neighborhood of $x = a$, let $f'(x)$ and $F'(x)$ be

continuous at all points except $x = a$. *If $f(x)$ and $F(x)$ approach infinity as x approaches a,*

$$\lim_{x \doteq a} \frac{f(x)}{F(x)} = \lim_{x \doteq a} \frac{f'(x)}{F'(x)}.$$

To show this let c be near a and on the same side as x. Since $f(x) - f(c)$ and $F(x) - F(c)$ are zero when $x = c$, by Theorem 1,

$$\frac{f'(x_1)}{F'(x_1)} = \frac{f(x) - f(c)}{F(x) - F(c)} = \frac{f(x)}{F(x)} \frac{1 - \dfrac{f(c)}{f(x)}}{1 - \dfrac{F(c)}{F(x)}},$$

where x_1 is between x and c. As x approaches a, $f(x)$ and $F(x)$ increase indefinitely. The quantities $f(c)/f(x)$ and $F(c)/F(x)$ approach zero. The right side of this equation therefore approaches

$$\lim_{x \doteq a} \frac{f(x)}{F(x)}.$$

Since x_1 is between c and a, by taking c sufficiently near to a the left side of the equation can be made to approach

$$\lim_{x \doteq a} \frac{f'(x)}{F'(x)}.$$

Since the two sides are always equal, we therefore conclude that

$$\lim_{x \doteq a} \frac{f(x)}{F(x)} = \lim_{x \doteq a} \frac{f'(x)}{F'(x)}.$$

Example 1. Find the value approached by $\dfrac{\sin x}{x}$ as x approaches zero.

Since the numerator and denominator are zero when $x = 0$, we can apply Theorem 2 and so get

$$\lim_{x \doteq 0} \frac{\sin x}{x} = \lim_{x \doteq 0} \frac{\cos x}{1} = 1.$$

Ex. 2. Find the value of $\displaystyle\lim_{x \doteq \pi} \frac{1 + \cos x}{(\pi - x)^2}$.

When $x = \pi$ the numerator and denominator are both zero. Hence

$$\lim_{x \doteq \pi} \frac{1 + \cos x}{(\pi - x)^2} = \lim_{x \doteq \pi} \frac{(-\sin x)}{-2(\pi - x)} = \frac{0}{0}.$$

Since this is indeterminate we apply the method a second time and so obtain

$$\lim_{x \doteq \pi} \frac{\sin x}{2(\pi - x)} = \lim_{x \doteq \pi} \frac{\cos x}{-2} = \frac{1}{2}.$$

The value required is therefore $\frac{1}{2}$.

Ex. 3. Find the value approached by $\dfrac{\tan 3x}{\tan x}$ as x approaches $\dfrac{\pi}{2}$.

When x approaches $\dfrac{\pi}{2}$ the numerator and denominator of this fraction approach ∞. Therefore, by Theorem 3,

$$\lim_{x \doteq \frac{\pi}{2}} \frac{\tan 3x}{\tan x} = \lim \frac{3 \sec^2 3x}{\sec^2 x} = \lim \frac{3 \cos^2 x}{\cos^2 3x}.$$

When x is replaced by $\dfrac{\pi}{2}$ the last expression takes the form $\dfrac{0}{0}$. Therefore

$$\lim \frac{3 \cos^2 x}{\cos^2 3x} = \lim \frac{6 \cos x \sin x}{6 \cos 3x \sin 3x}$$

$$\doteq \lim \frac{\cos^2 x - \sin^2 x}{3(\cos^2 3x - \sin^2 3x)} = \frac{1}{3}.$$

74. The Forms $0 \cdot \infty$ and $\infty - \infty$. — By transforming the expression to a fraction it will take the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.

For example,

$$x \ln x$$

has the form $0 \cdot \infty$ when $x = 0$. It can, however, be written

$$x \ln x = \frac{\ln x}{\frac{1}{x}},$$

which has the form $\frac{\infty}{\infty}\cdot$

The expression

$$\sec x - \tan x$$

has the form $\infty - \infty$ when $x = \frac{\pi}{2}\cdot$ It can, however, be written

$$\sec x - \tan x = \frac{1}{\cos x} - \frac{\sin x}{\cos x} = \frac{1 - \sin x}{\cos x},$$

which becomes $\frac{0}{0}$ when $x = \frac{\pi}{2}\cdot$

75. The Forms 0°, 1^∞, ∞°. — The logarithm of the given function has the form $0 \cdot \infty$. From the limit of the logarithm the limit of the function can be determined.

Example. Find the limit of $(1 + x)^{\frac{1}{x}}$ as x approaches zero.

Let $$y = (1 + x)^{\frac{1}{x}}.$$

Then

$$\ln y = \frac{1}{x} \ln (1 + x) = \frac{\ln (1 + x)}{x}.$$

When x is zero this last expression becomes $\frac{0}{0}$. Therefore

$$\lim_{x \doteq 0} \frac{\ln (1 + x)}{x} = \lim \frac{1}{1 + x} = 1.$$

The limit of $\ln y$ being 1, the limit of y is e.

EXERCISES

1. Show by Rolle's Theorem that the equation

$$x^4 - 4x - 1 = 0$$

cannot have more than two real roots.

Determine the values of the following limits:

2. $\displaystyle \lim_{x \doteq 1} \frac{x^9 - 1}{x^{10} - 1}.$

3. $\displaystyle \lim_{x \doteq 1} \frac{x^n - 1}{x - 1}.$

4. $\displaystyle \lim_{x \doteq 0} \frac{1 - \cos x}{\sin x}.$

5. $\displaystyle \lim_{x \doteq a} \frac{e^x - e^a}{x - a}.$

6. $\displaystyle \lim_{x \doteq 0} \frac{\tan x - x}{x - \sin x}.$

7. $\displaystyle \lim_{x \doteq 0} \frac{x^2 \cos x}{\cos x - 1}.$

8. $\displaystyle \lim_{x \doteq 3} \frac{\ln (x - 2)}{x - 3}.$

9. $\displaystyle \lim_{x \doteq 0} \frac{\ln \cos x}{x}.$

10. $\displaystyle \lim_{x \doteq 2} \frac{\sin^2 \pi x}{(x - 2)^2}.$

11. $\displaystyle \lim_{x \doteq \frac{\pi}{2}} \frac{1 + \cos x - \sin x}{\cos x\,(2 \sin x - 1)}.$

12. $\displaystyle \lim_{x \doteq \alpha} \frac{\log_{10} (\sin x - \sin \alpha)}{\log_{10} (\tan x - \tan \alpha)}.$

13. $\displaystyle \lim_{x \doteq 0} \frac{6 \sin x - 6x + x^3}{x^2}.$

14. $\displaystyle \lim_{\phi \doteq \frac{\pi}{4}} \frac{\sec^2 \phi - 2 \tan \phi}{1 + \cos 4\phi}.$

15. $\displaystyle \lim_{x \doteq 0} \frac{\ln x}{\cot x}.$

16. $\displaystyle \lim_{x = \infty} \frac{\ln x}{x}.$

17. $\displaystyle \lim_{x \doteq \frac{\pi}{2}} \frac{\sec 3x - x}{\sec x - x}.$

18. $\displaystyle \lim_{x \doteq \frac{\pi}{4}} \frac{1 + \tan 2x}{\sec\left(x + \frac{\pi}{4}\right)}.$

19. $\displaystyle \lim_{x \doteq 0} x \cot x.$

20. $\displaystyle \lim_{x \doteq \frac{\pi}{2}} \tan x \cos 3x.$

21. $\displaystyle \lim_{x = \infty} (x + a) \ln \left(1 + \frac{a}{x}\right).$

22. $\displaystyle \lim_{x \doteq 3} (x - 3) \cot (\pi x).$

23. $\displaystyle \lim_{n = \infty} n \left[f\left(x + \frac{dx}{n}\right) - f(x)\right].$

24. $\displaystyle \lim_{x \doteq 0} x^2 e^{\frac{1}{x^2}}.$

25. $\displaystyle \lim_{x \doteq 0} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right).$

26. $\displaystyle \lim_{x \doteq 0} (\cot x - \ln x).$

27. $\displaystyle \lim_{x \doteq \frac{\pi}{2}} \left[\tan x - \frac{1}{\sin x - \sin^2 x}\right].$

28. $\displaystyle \lim_{x \doteq 0} x^x.$

29. $\displaystyle \lim_{x \doteq \frac{\pi}{2}} (\sin x)^{\tan x}.$

30. $\displaystyle \lim_{x \doteq 0} (1 + ax)^{\frac{1}{x}}.$

31. $\displaystyle \lim_{x = \infty} (x^m - a^m)^{\frac{1}{\ln x}}.$

CHAPTER X

SERIES AND APPROXIMATIONS

76. Mean Value Theorem. — *If* $f(x)$ and $f'(x)$ *are continuous from* $x = a$ *to* $x = b$, *there is a value* x_1 *between* a *and* b *such that*

$$\frac{f(b) - f(a)}{b - a} = f'(x_1). \tag{76}$$

To show this consider the curve $y = f(x)$. Since $f(a)$ and $f(b)$ are the ordinates at $x = a$ and $x = b$,

$$\frac{f(b) - f(a)}{b - a} = \text{slope of chord } AB.$$

On the arc AB let P_1 be a point at maximum distance from

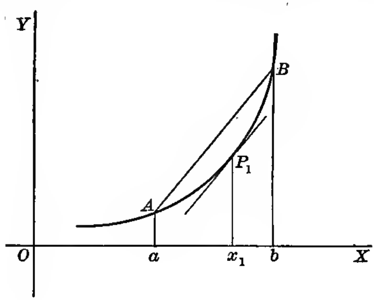

FIG. 76.

the chord. The tangent at P_1 will be parallel to the chord and so its slope $f'(x_1)$ will equal that of the chord. Therefore

$$\frac{f(b) - f(a)}{b - a} = f'(x_1),$$

which was to be proved.

Replacing b by x and solving for $f(x)$, equation (76) becomes

$$f(x) = f(a) + (x - a) f'(x_1),$$

x_1 being between a and x. This is a special case of a more general theorem which we shall now prove.

77. Taylor's Theorem. — *If $f(x)$ and all its derivatives used are continuous from a to x, there is a value x_1 between a and x such that*

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a)$$
$$+ \frac{(x-a)^3}{3!} f'''(a) + \cdots + \frac{(x-a)^n}{n!} f^n(x_1).$$

To prove this let

$$\phi(x) = f(x) - f(a) - (x-a) f'(a)$$
$$- \frac{(x-a)^2}{2!} f''(a) - \cdots - \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a).$$

It is easily seen that

$$\phi(a) = 0, \quad \phi'(a) = 0, \quad \phi''(a) = 0,$$
$$\ldots \phi^{n-1}(a) = 0, \quad \phi^n(x) = f^n(x).$$

When $x = a$ the function

$$\frac{\phi(x)}{(x-a)^n}$$

therefore assumes the form $\frac{0}{0}$. By Art. 73 there is then a value z_1 between a and x such that

$$\frac{\phi(x)}{(x-a)^n} = \frac{\phi'(z_1)}{n(z_1-a)^{n-1}}.$$

This new expression becomes $\frac{0}{0}$ when $z_1 = a$. There is consequently a value z_2 between z_1 and a (and so between x and a) such that

$$\frac{\phi'(z_1)}{n(z_1-a)^{n-1}} = \frac{\phi''(z_2)}{n(n-1)(z_2-a)^{n-2}}.$$

A continuation of this argument gives finally

$$\frac{\phi(x)}{(x-a)^n} = \frac{\phi^n(z_n)}{n!} = \frac{f^n(z_n)}{n!},$$

z_n being between x and a. If $x_1 = z_n$ we then have

$$\phi(x) = \frac{(x-a)^n}{n!} f^n(x_1).$$

Equating this to the original value of $\phi(x)$ and solving for $f(x)$, we get

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^n}{n!} f^n(x_1),$$

which was to be proved.

Example. Prove

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4\,x_1^4},$$

where x_1 is between 1 and x.

When $x = 1$ the values of $\ln x$ and its derivatives are

$$\begin{aligned}
f(x) &= \ln(x), & f(1) &= 0,\\
f'(x) &= \frac{1}{x}, & f'(1) &= 1,\\
f''(x) &= -\frac{1}{x^2}, & f''(1) &= -1,\\
f'''(x) &= \frac{2}{x^3}, & f'''(1) &= 2,\\
f''''(x) &= -\frac{6}{x^4}, & f''''(x_1) &= -\frac{6}{(x_1)^4}.
\end{aligned}$$

Taking $a = 1$, Taylor's Theorem gives

$$\ln x = 0 + 1(x-1) - \frac{1}{2}(x-1)^2 + \frac{2}{6}(x-1)^3 - \frac{6}{24}\,\frac{(x-1)^4}{x_1^4},$$

which is the result required.

78. Approximate Values of Functions. — The last term in Taylor's formula

$$\frac{(x-a)^n}{n!} f^n(x_1) = R_n$$

is called the *remainder*. If this is small, an approximate value of the function is

$$f(x) = f(a) + (x - a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a),$$

the error in the approximation being equal to the remainder.

To compute $f(x)$ by this formula, we must know the values of $f(a)$, $f'(a)$, etc. *We must then assign a value to a such that $f(a)$, $f'(a)$, etc., are known. Furthermore, a should be as close as possible to the value x at which $f(x)$ is wanted.* For, the smaller $x - a$, the fewer terms $(x - a)^2$, $(x - a)^3$, etc., need be computed to give a required approximation.

Example 1. Find tan 46° to four decimals.

The value closest to 46° for which $\tan x$ and its derivatives are known is 45°. Therefore we let $a = \frac{\pi}{4}$.

$$f(x) = \tan x, \qquad f\left(\frac{\pi}{4}\right) = 1,$$

$$f'(x) = \sec^2 x, \qquad f'\left(\frac{\pi}{4}\right) = 2,$$

$$f''(x) = 2 \sec^2 x \tan x, \qquad f''\left(\frac{\pi}{4}\right) = 4,$$

$$f'''(x) = 2 \sec^4 x + 4 \sec^2 x \tan^2 x.$$

Using these values in Taylor's formula, we get

$$\tan x = 1 + 2\left(x - \frac{\pi}{4}\right) + \frac{4}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{f'''(x_1)}{3!}\left(x - \frac{\pi}{4}\right)^3$$

and

$$\tan 46° = 1 + 2\left(\frac{\pi}{180}\right) + 2\left(\frac{\pi}{180}\right)^2 = 1.0355$$

approximately. Since x_1 is between 45° and 46°, $f'''(x_1)$ does not differ much from

$$f'''(45°) = 8 + 8 = 16.$$

The error in the above approximation is thus very nearly

$$\frac{16}{6}\left(\frac{\pi}{180}\right)^3 < \frac{8}{3\,(50)^3} < \frac{1}{40{,}000} = 0.000025.$$

It is therefore correct to 4 decimals.

Ex. 2. Find the value of e to four decimals.

The only value of x for which e^x and its derivatives are known is $x = 0$. We therefore let a be zero.

$$f(x) = e^x,\quad f'(x) = e^x,\quad f''(x) = e^x,\quad \ldots\ldots,\quad f^n(x) = e^x,$$
$$f(0) = 1,\quad f'(0) = 1,\quad f''(0) = 1,\quad \ldots\ldots,\quad f^n(x_1) = e^{x_1}.$$

By Taylor's Theorem,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n e^{x_1}}{n!}.$$

Letting $x = 1$, this becomes

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!} + \frac{e^{x_1}}{n!}.$$

In particular, if $n = 2$,

$$e = 2 + \tfrac{1}{2}\,e^{x_1}.$$

Since x_1 is between 0 and 1, e is then between $2\frac{1}{2}$ and $2 + \frac{1}{2}\,e$, and therefore between $2\frac{1}{2}$ and 4. To get a better approximation let $n = 9$. Then

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{8!} = 2.7183$$

approximately, the error being

$$\frac{e^{x_1}}{9!} \leqq \frac{e}{9!} \leqq \frac{4}{9!} < .00002.$$

The value 2.7183 is therefore correct to four decimals.

EXERCISES

Determine the values of the following functions correct to four decimals:

1. $\sin 5°$.
2. $\cos 32°$.
3. $\cot 43°$.
4. $\tan 58°$.
5. $\sec (10°)$.
6. $\ln\left(\frac{9}{10}\right)$.
7. $\sqrt{e}$.
8. $\tan^{-1}\left(\frac{1}{10}\right)$.
9. Given $\ln 3 = 1.0986$, $\ln 5 = 1.6094$, find $\ln 17$.

79. Taylor's and Maclaurin's Series. — As n increases indefinitely, the remainder in Taylor's formula

$$R_n = \frac{(x-a)^n}{n} f^n(x_1)$$

often approaches zero. In that case

$$f(x) = \lim_{n=\infty}\left[f(a) + (x-a)f'(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a)\right].$$

This is usually written

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots,$$

the dots at the end signifying the limit of the sum as the number of terms is indefinitely increased. Such an infinite sum is called an *infinite series.* This one is called *Taylor's Series.*

In particular, if $a = 0$, Taylor's Series becomes

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots.$$

This is called *Maclaurin's Series.*

Example. Show that $\cos x$ is represented by the series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

The series given contains powers of x. This happens when $a = 0$, that is, when Taylor's Series reduces to Maclaurin's.

$$\begin{aligned}
f(x) &= \cos x, & f(0) &= 1,\\
f'(x) &= -\sin x, & f'(0) &= 0,\\
f''(x) &= -\cos x, & f''(0) &= -1,\\
f'''(x) &= \sin x, & f'''(0) &= 0,\\
f''''(x) &= \cos x, & f''''(0) &= 1.
\end{aligned}$$

These values give

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \pm \frac{x^n}{n!} f^n(x_1).$$

The nth derivative of $\cos x$ is $\pm\cos x$ or $\pm\sin x$, depending on whether n is even or odd. Since $\sin x$ and $\cos x$ are never greater than 1, $f_n(x_1)$ is not greater than 1. Furthermore

$$\frac{x^n}{n!} = \frac{x}{1}\cdot\frac{x}{2}\cdot\frac{x}{3}\cdot\ \cdots\ \cdot\frac{x}{n}$$

can be made as small as you please by taking n sufficiently large. Hence the remainder approaches zero and so

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots,$$

which was to be proved.

EXERCISES

1. $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots.$
2. $\cos x = \dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}}\left(x - \dfrac{\pi}{4}\right) - \dfrac{1}{2\sqrt{2}}\left(x - \dfrac{\pi}{4}\right)^2 + \dfrac{1}{6\sqrt{2}}\left(x - \dfrac{\pi}{4}\right)^3 + \cdots$
3. $2^x = 1 + x\ln 2 + \dfrac{(x\ln 2)^2}{2!} + \dfrac{(x\ln 2)^3}{3!} + \cdots.$
4. $e^x \sin x = x + 2\cdot\dfrac{x^2}{2!} + 2\dfrac{x^3}{3!} - 4\dfrac{x^5}{5!} - 8\dfrac{x^6}{6!} - \dfrac{8x^7}{7!} + \cdots.$
5. $e^x \cos x = 1 + x - \dfrac{2x^3}{3!} - \dfrac{4x^4}{4!} - \dfrac{4x^5}{5!} + \dfrac{8x^7}{7!} + \cdots.$
6. $(a + x)^n = a^n + na^{n-1}x + \dfrac{n(n-1)}{2!}a^{n-2}x^2 + \cdots$, if $|x|$* $< |a|$.
7. $\sqrt{x} = 2 + \dfrac{x-4}{4} - \dfrac{(x-4)^2}{64} + \dfrac{(x-4)^3}{512} - \cdots$, if $|x - 4| < 1$.
8. $\ln x = \ln 3 + \dfrac{x-3}{3} - \dfrac{(x-3)^2}{2\cdot 3^2} + \dfrac{(x-3)^3}{3\cdot 3^3} - \cdots$, if $|x - 3| < 1$.
9. $\ln(x + 5) = \ln 6 + \dfrac{x-1}{6} - \dfrac{(x-1)^2}{2\cdot 6^2} + \dfrac{(x-1)^3}{3\cdot 6^3} - \cdots$, if $|x-1| < 1$.

80. Convergence and Divergence of Series.— An infinite series is said to *converge* if the sum of the first n terms approaches a limit as n increases indefinitely. If this sum does not approach a limit, the series is said to *diverge*.

The series for $\sin x$ and $\cos x$ converge for all values of x. The geometrical series

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots$$

* The symbol $|x|$ is used to represent the numerical value of x without its algebraic sign. Thus, $|-3| = |3| = 3$.

converges when r is numerically less than 1. For the sum of the first n terms is

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = a\,\frac{1 - r^n}{1 - r}.$$

If r is numerically less than 1, r^n approaches zero and S_n approaches

$$S = \frac{a}{1 - r}$$

as n increases indefinitely.

The series

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

is divergent, for the sum oscillates between 0 and 1 and does not approach a limit. The geometrical series

$$1 + 2 + 4 + 8 + 16 + \cdots$$

diverges because the sum increases indefinitely and so does not approach a limit.

81. Tests for Convergence. — The convergence of a series can often be determined from the problem in which it occurs. Thus the series

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

converges because the sum of n terms approaches $\cos x$ as n increases indefinitely.

The terms near the beginning of a series (if they are all finite) have no influence on the convergence or divergence of the series. This is determined by terms indefinitely far out in the series.

82. General Test. — *For the series*

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

to converge it is necessary and sufficient that the sum of terms beyond u_n approach zero as n increases indefinitely.

For, if the series converges, the sum of n terms must approach a limit and so the sum of terms beyond the nth must approach zero.

83. Comparison Test. — *A series is convergent if beyond a certain point its terms are in numerical value respectively less than those of a convergent series whose terms are all positive.*

For, if a series converges, the sum of terms beyond the nth will approach zero as n increases indefinitely. If then another series has lesser corresponding terms, their sum will approach zero and the series will converge.

84. Ratio Test. — *If the ratio* $\frac{u_{n+1}}{u_n}$ *of consecutive terms approaches a limit* r *as* n *increases indefinitely, the series*

$$u_1 + u_2 + u_3 + \cdots + u_n + u_{n+1} + \cdots$$

is convergent if r *is numerically less than 1 and divergent if* r *is numerically greater than 1.*

Since the limit is r, by taking n sufficiently large the ratio of consecutive terms can be made as nearly r as we please. If $r < 1$, let r_1 be a fixed number between r and 1. We can take n so large that the ratio of consecutive terms is less than r_1. Then

$$u_{n+1} < r_1 u_n, \quad u_{n+2} < r_1 u_{n+1} < r_1^2 u_n, \quad \text{etc.}$$

Beyond u_n the terms of the given series are therefore less than those of the geometrical progression

$$u_n + r_1 u_n + r_1^2 u_n + \cdots$$

which converges since r_1 is numerically less than 1. Consequently the given series converges.

If, however, r is greater than 1, the terms of the series must ultimately increase. The terms do not then approach zero and their sum cannot approach a limit.

Example. Find for what values of x the series

$$x + 2x^2 + 3x^3 + 4x^4 + \cdots$$

converges.

The ratio of consecutive terms is

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)x^{n+1}}{nx^n} = \left(1 + \frac{1}{n}\right)x.$$

The limit of this ratio is

$$r = \lim_{n=\infty} \left(1 + \frac{1}{n}\right) x = x.$$

The series will converge if x is numerically less than 1.

85. Power Series. — A series of powers of $(x - a)$ of the form

$$P(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots,$$

where a, a_0, a_1, a_2, etc., are constants, is called a *power series.*

If a power series converges when $x = b$, it will converge for all values of x nearer to a than b is, that is, such that

$$|x - a| < |b - a|.$$

In fact, if the series converges when $x = b$, each term of

$$a_0 + a_1(b-a) + a_2(b-a)^2 + a_3(b-a)^3 + \cdots.$$

will be less than a maximum value M, that is,

$$|a_n(b-a)^n| < M.$$

Consequently,

$$|a_n| < \frac{M}{|(b-a)^n|}.$$

The terms of the series

$$a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots$$

are then respectively less than those of the geometrical series

$$M + \frac{M}{|b-a|}|x-a| + \frac{M}{|b-a|^2}|x-a|^2 + \frac{M}{|b-a|^3}|x-a|^3 + \cdots$$

in which the ratio is

$$\frac{|x-a|}{|b-a|}.$$

If then $|x - a| < |b - a|$, the progression and consequently the given series will converge.

If a power series diverges when $x = b$, it will diverge for all values of x further from a than b is, that is, such that

$$|x - a| > |b - a|.$$

For it could not converge beyond b, since by the proof just given it would then converge at b.

This theorem shows in certain cases why a Taylor's Series is not convergent. Take, for example, the series

$$\ln(1+x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots .$$

As x approaches -1, $\ln(1+x)$ approaches infinity. Since a convergent series cannot have an infinite value, we should expect the series to diverge when $x = -1$. It must then diverge when x is at a distance greater than 1 from $a = 0$. The series in fact converges between $x = -1$ and $x = 1$ and diverges for values of x numerically greater than 1.

86. Operations with Power Series. — It is shown in more advanced treatises that convergent series can be added, subtracted, multiplied and divided like polynomials. In case of division, however, the resulting series will not usually converge beyond a point where the denominator is zero.

Example. Express $\tan x$ as a series in powers of x.

We could use Maclaurin's series with $f(x) = \tan x$. It is easier, however, to expand $\sin x$ and $\cos x$ and divide the one by the other to get $\tan x$. Thus

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots}{1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots} = x + \frac{x^3}{3} + \frac{2\,x^5}{15} + \cdots .$$

EXERCISES

1. Show that

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right),$$

and that the series converges when $|x| < 1$.

2. By expanding $\cos 2\,x$, show that

$$\sin^2 x = \frac{1 - \cos 2\,x}{2} = 2\,\frac{x^2}{2!} - 2^3\,\frac{x^4}{4!} + 2^5\,\frac{x^6}{6!} - \cdots .$$

Prove that the series converges for all values of x.

3. Show that

$$(1 + e^x)^2 = 1 + 2e^x + e^{2x} = 4 + 4x + 3x^2 + \tfrac{5}{3}x^3 + \tfrac{3}{4}x^4 + \cdots$$

and that the series converges for all values of x.

4. Given $f(x) = \sin^{-1} x$, show that

$$f'(x) = \frac{1}{\sqrt{1 - x^2}} = (1 - x^2)^{-\frac{1}{2}}.$$

Expand this by the binomial theorem and determine $f''(x)$, etc., by differentiating the result. Hence show that

$$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\frac{x^5}{5} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\frac{x^7}{7} + \cdots$$

and that the series converges when $|x| < 1$.

5. By a method similar to that used in Ex. 4, show that

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

and that the series converges when $|x| < 1$.

6. Prove

$$\sec x = \frac{1}{\cos x} = 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \cdots.$$

For what values of x do you think the series converges?

CHAPTER XI

PARTIAL DIFFERENTIATION

87. Functions of Two or More Variables. — A quantity u is called a function of two independent variables x and y,

$$u = f(x, y),$$

if u is determined when arbitrary values (or values arbitrary within certain limits) are assigned to x and y.

For example,

$$u = \sqrt{1 - x^2 - y^2}$$

is a function of x and y. If u is to be real, x and y must be so chosen that $x^2 + y^2$ is not greater than 1. Within that limit, however, x and y can be chosen independently and a value of u will then be determined.

In a similar way we define a function of three or more independent variables. An illustration of a function of variables that are not independent is furnished by the area of a triangle. It is a function of the sides a, b, c and angles A, B, C of the triangle, but is not a function of these six quantities considered as independent variables; for, if values not belonging to the same triangle are given to them, no triangle and consequently no area will be determined.

The increment of a function of several variables is its increase when all the variables change. Thus, if

$$u = f(x, y),$$
$$u + \Delta u = f(x + \Delta x, y + \Delta y)$$

and so

$$\Delta u = f(x + \Delta x, y + \Delta y) - f(x, y).$$

A function is called *continuous* if its increment approaches zero when all the increments of the variables approach zero.

88. Partial Derivatives. — Let

$$u = f(x, y)$$

be a function of two independent variables x and y. If we keep y constant, u is a function of x. The derivative of this function with respect to x is called the *partial derivative* of u with respect to x and is denoted by

$$\frac{\partial u}{\partial x} \quad \text{or} \quad f_x(x, y).$$

Similarly, if we differentiate with respect to y with x constant, we get the partial derivative with respect to y denoted by

$$\frac{\partial u}{\partial y} \quad \text{or} \quad f_y(x, y).$$

For example, if

$$u = x^2 + xy - y^2,$$

then

$$\frac{\partial u}{\partial x} = 2x + y, \quad \frac{\partial u}{\partial y} = x - 2y.$$

Likewise, if u is a function of any number of independent variables, the partial derivative with respect to one of them is obtained by differentiating with the others constant.

89. Higher Derivatives. — The first partial derivatives are functions of the variables. By differentiating these functions partially, we get higher partial derivatives.

For example, the derivatives of $\frac{\partial u}{\partial x}$ with respect to x and y are

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y\, \partial x}.$$

Similarly,

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x\, \partial y}, \quad \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2}.$$

It can be shown that

$$\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial^2 u}{\partial y\, \partial x},$$

if both derivatives are continuous, that is, *partial derivatives are independent of the order in which the differentiations are performed.**

Example. $u = x^2y + xy^2.$

$$\frac{\partial u}{\partial x} = 2\,xy + y^2, \qquad \frac{\partial u}{\partial y} = x^2 + 2\,xy,$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(2\,xy + y^2) = 2\,y, \quad \frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial}{\partial y}(2\,xy + y^2) = 2\,x + 2\,y,$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial}{\partial x}(x^2 + 2\,xy) = 2\,x + 2\,y, \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(x^2 + 2\,xy) = 2\,x.$$

90. Dependent Variables. — It often happens that some of the variables are functions of others. For example, let

$$u = x^2 + y^2 + z^2$$

and let z be a function of x and y. When y is constant, z will be a function of x and the partial derivative of u with respect to x will be

$$\frac{\partial u}{\partial x} = 2\,x + 2\,z\frac{\partial z}{\partial x}.$$

Similarly, the partial derivative with respect to y with x constant is

$$\frac{\partial u}{\partial y} = 2\,y + 2\,z\frac{\partial z}{\partial y}.$$

If, however, we consider z constant, the partial derivatives are

$$\frac{\partial u}{\partial x} = 2\,x, \quad \frac{\partial u}{\partial y} = 2\,y.$$

The value of a partial derivative thus depends on what quantities are kept constant during the differentiation.

The quantities kept constant are sometimes indicated by subscripts. Thus, in the above example

$$\left(\frac{\partial u}{\partial x}\right)_{y,z} = 2\,x, \quad \left(\frac{\partial u}{\partial x}\right)_{y} = 2\,x + 2\,z\frac{\partial z}{\partial x}, \quad \left(\frac{\partial u}{\partial x}\right)_{z} = 2\,x + 2\,y\frac{\partial y}{\partial x}.$$

* For a proof see Wilson, *Advanced Calculus*, § 50.

It will usually be clear from the context what independent variables u is considered a function of. Then $\frac{\partial u}{\partial x}$ will represent the derivative with all those variables except x constant.

Example. If a is a side and A the opposite angle of a right triangle with hypotenuse c, find $\left(\frac{\partial a}{\partial c}\right)_A$.

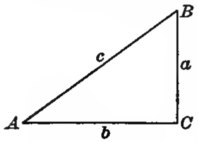

FIG. 90.

From the triangle it is seen that

$$a = c \sin A.$$

Differentiating with A constant, we get

$$\frac{\partial a}{\partial c} = \sin A,$$

which is the value required.

91. Geometrical Representation. — Let $z = f(x, y)$ be the equation of a surface. The points with constant y-coördinate form the curve AB (Fig. 91a) in which the plane $y =$ constant intersects the surface. In this plane z is the vertical and x the horizontal coördinate. Consequently,

$$\frac{\partial z}{\partial x}$$

is the slope of the curve AB at P.

Similarly, the locus of points with given x is the curve CD and

$$\frac{\partial z}{\partial y}$$

is the slope of this curve at P.

Example. Find the lowest point on the paraboloid

$$z = x^2 + y^2 - 2x - 4y + 6.$$

At the lowest point, the curves AB and CD (Fig. 91b) will have horizontal tangents. Hence

$$\frac{\partial z}{\partial x} = 2x - 2 = 0, \qquad \frac{\partial z}{\partial y} = 2y - 4 = 0.$$

Consequently, $x = 1$, $y = 2$. These values substituted in the equation of the surface give $z = 1$. The point required is then $(1, 2, 1)$. That this is really the lowest point is shown by the graph.

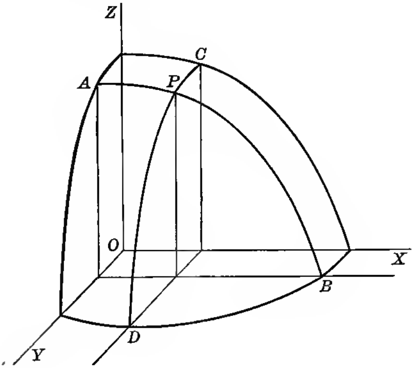

FIG. 91a.

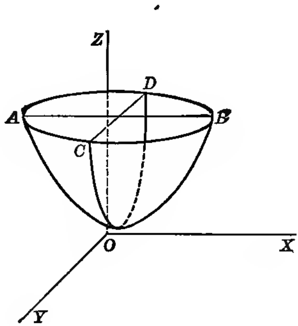

FIG. 91b.

EXERCISES

In each of the following exercises show that the partial derivatives satisfy the equation given:

1. $u = \dfrac{x^2 + y^2}{x + y}$, $\qquad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = u.$

2. $z = (x + a)(y + b)$, $\qquad \dfrac{\partial z}{\partial x}\dfrac{\partial z}{\partial y} = z.$

3. $z = (x^2 + y^2)^n$, $\qquad y\dfrac{\partial z}{\partial x} = x\dfrac{\partial z}{\partial y}.$

4. $u = \ln(x^2 + xy + y^2)$, $\qquad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2.$

5. $u = \dfrac{y}{z} + \dfrac{z}{x} + \dfrac{x}{y}$, $\qquad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0.$

6. $u = \tan^{-1}\left(\dfrac{y}{x}\right)$, $\qquad \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0.$

7. $u = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$, $\qquad \dfrac{\partial^2 u}{dx^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0.$

In each of the following exercises verify that

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}.$$

8. $u = \dfrac{y}{x}$.

9. $u = \ln (x^2 + y^2)$.

10. $u = \sin (x + y)$.

11. $u = xyz$.

12. Given $v = \sqrt{x^2 + y^2 + z^2}$, verify that

$$\frac{\partial^3 v}{\partial x\, \partial y\, \partial z} = \frac{\partial^3 v}{\partial z\, \partial y\, \partial x}.$$

Prove the following relations assuming that z is a function of x and y:

13. $u = (x + z)\, e^{y+z}$, $\quad \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = (1 + x + z)\left(1 + \dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y}\right) e^{y+z}$.

14. $u = xyz$, $\quad z\left(x \dfrac{\partial u}{\partial x} - y \dfrac{\partial u}{\partial y}\right) = u\left(x \dfrac{\partial z}{\partial x} - y \dfrac{\partial z}{\partial y}\right)$.

15. $u = e^x + e^y + e^z$, $\quad \dfrac{\partial^2 u}{\partial x\, \partial y} = e^z\left(\dfrac{\partial^2 z}{\partial x\, \partial y} + \dfrac{\partial z}{\partial x} \dfrac{\partial z}{\partial y}\right)$.

16. $\dfrac{\partial}{\partial x}\left(z \dfrac{\partial u}{\partial x} - u \dfrac{\partial z}{\partial x}\right) = z \dfrac{\partial^2 u}{\partial x^2} - u \dfrac{\partial^2 z}{\partial x^2}$.

17. If $x = r \cos \theta$, $y = r \sin \theta$, show that

$$\left(\frac{\partial x}{\partial r}\right)_\theta = \left(\frac{\partial r}{\partial x}\right)_y.$$

18. Let a and b be the sides of a right triangle with hypotenuse c and opposite angles A and B. Let p be the perpendicular from the vertex of the right angle to the hypotenuse. Show that

$$\left(\frac{\partial p}{\partial a}\right)_b = \frac{b^3}{c^3}, \qquad \left(\frac{\partial p}{\partial a}\right)_A = \frac{b}{c}.$$

19. If K is the area of a triangle, a side and two adjacent angles of which are c, A, B, show that

$$\left(\frac{\partial K}{\partial a}\right)_{c,\, B} = \frac{b^2}{2}, \qquad \left(\frac{\partial K}{\partial B}\right)_{c,\, A} = \frac{a^2}{2}.$$

20. If K is the area of a triangle with sides a, b, c, show that

$$\left(\frac{\partial K}{\partial a}\right)_{b,\, c} = \frac{a}{2} \cot A.$$

21. Find the lowest point on the surface

$$z = 2\, x^2 + y^2 + 8\, x - 2\, y + 9.$$

22. Find the highest point on the surface

$$z = 2\, y - x^2 + 2\, xy - 2\, y^2 + 1.$$

92. Increment. — Let $u = f\,(x, y)$ be a function of two independent variables x and y. When x changes to $x + \Delta x$ and y to $y + \Delta y$, the increment of u is

$$\Delta u = f\,(x + \Delta x,\, y + \Delta y) - f\,(x, y). \tag{92a}$$

By the mean value theorem, Art. 76,

$$f(x + \Delta x, y + \Delta y) = f(x, y + \Delta y) + \Delta x f_x(x_1, y + \Delta y),$$

x_1 lying between x and $x + \Delta x$. Similarly

$$f(x, y + \Delta y) = f(x, y) + \Delta y f_y(x, y_1),$$

y_1 being between y and $y + \Delta y$. Using these values in (92a), we get

$$\Delta u = \Delta x f_x(x_1, y + \Delta y) + \Delta y f_y(x, y_1). \tag{92b}$$

As Δx and Δy approach zero, x_1 approaches x and y_1 approaches y. If $f_x(x, y)$ and $f_y(x, y)$ are continuous,

$$f_x(x_1, y + \Delta y) = f_x(x, y) + \epsilon_1 = \frac{\partial u}{\partial x} + \epsilon_1,$$

$$f_y(x, y_1) = f_y(x, y) + \epsilon_2 = \frac{\partial u}{\partial y} + \epsilon_2,$$

ϵ_1 and ϵ_2 approaching zero as Δx and Δy approach zero. These values substituted in (92b) give

$$\Delta u = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y. \tag{92c}$$

The quantity

$$\frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y$$

is called the *principal part* of Δu. It differs from Δu by an amount $\epsilon_1 \Delta x + \epsilon_2 \Delta y$. As Δx and Δy approach zero, ϵ_1 and ϵ_2 approach zero and so this difference becomes an indefinitely small fraction of the larger of the increments Δx and Δy. We express this by saying the principal part differs from Δu by an infinitesimal of higher order than Δx and Δy (Art. 9). When Δx and Δy are sufficiently small this principal part then gives a satisfactory approximation for Δu.

Analogous results can be obtained for any number of independent variables. For example, if there are three independent variables x, y, z, the principal part of Δu is

$$\frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \frac{\partial u}{\partial z}\Delta z.$$

In each case, if the partial derivatives are continuous, the principal part differs from Δu by an amount which becomes indefinitely small in comparison with the largest of the increments of the independent variables as those increments all approach zero.

Example. Find the change in the volume of a cylinder when its length increases from 6 ft. to 6 ft. 1 in. and its diameter decreases from 2 ft. to 23 in.

Since the volume is $v = \pi r^2 h$, the exact change is

$$\Delta v = \pi \left(1 - \tfrac{1}{24}\right)^2 \left(6 + \tfrac{1}{12}\right) - \pi \cdot 1^2 \cdot 6 = -0.413\,\pi \text{ cu. ft.}$$

The principal part of this increment is

$$\frac{\partial v}{\partial r}\,\Delta r + \frac{\partial v}{\partial h}\,\Delta h = 2\,\pi\, rh\left(-\frac{1}{24}\right) + \pi\, r^2\left(\frac{1}{12}\right) = -0.417\,\pi \text{ cu. ft.}$$

93. Total Differential. — If u is a function of two independent variables x and y, the *total differential* of u is the principal part of Δu, that is,

$$du = \frac{\partial u}{\partial x}\,\Delta x + \frac{\partial u}{\partial y}\,\Delta y. \tag{93a}$$

This definition applies to any function of x and y. The particular values $u = x$ and $u = y$ give

$$dx = \Delta x, \qquad dy = \Delta y, \tag{93b}$$

that is, *the differentials of the independent variables are equal to their increments.*

Combining (93a) and (93b), we get

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy. \tag{93c}$$

We shall show later (Art. 97) that this equation is valid even if x and y are not the independent variables.

The quantities

$$d_x u = \frac{\partial u}{\partial x}\,dx, \qquad d_y u = \frac{\partial u}{\partial y}\,dy$$

are called *partial differentials.* Equation (93c) expresses that *the total differential of a function is equal to the sum of the partial differentials obtained by letting the variables change one at a time.*

Similar results can be obtained for functions of any number of variables. For instance, if u is a function of three independent variables x, y, z,

$$du = \frac{\partial u}{\partial x}\,\Delta x + \frac{\partial u}{\partial y}\,\Delta y + \frac{\partial u}{\partial z}\,\Delta z.$$

The particular values $u = x$, $u = y$, $u = z$ give

$$dx = \Delta x, \qquad dy = \Delta y, \qquad dz = \Delta z.$$

The previous equation can then be written

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy + \frac{\partial u}{\partial z}\,dz \tag{93d}$$

and in this form it can be proved valid even when x, y, z are not the independent variables.

Example 1. Find the total differential of the function

$$u = x^2y + xy^2.$$

By equation (93c)

$$\begin{aligned} du &= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \\ &= (2\,xy + y^2)\,dx + (x^2 + 2\,xy)\,dy. \end{aligned}$$

Ex. 2. Find the error in the volume of a rectangular box due to small errors in its three edges.

Let the edges be x, y, z. The volume is then

$$v = xyz.$$

The error in v, due to small errors Δx, Δy, Δz in x, y, z, is Δv. If the increments are sufficiently small, this will be approximately

$$dv = yz\,dx + xz\,dy + xy\,dz.$$

Dividing by v, we get

$$\begin{aligned} \frac{dv}{v} &= \frac{yz\,dx + xz\,dy + xy\,dz}{xyz} \\ &= \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}. \end{aligned}$$

Now $\frac{dx}{x}$ expresses the error dx as a fraction or percentage of x.

The equation just obtained expresses that the percentage error in the volume is equal to the sum of the percentage errors in the edges. If, for example, the error in each edge is not more than one per cent, the error in the volume is not more than three per cent.

94. Calculation of Differentials. — In proving the formulas of differentiation it was assumed that u, v, etc., were functions of a single variable. It is easy to show that the same formulas are valid when those quantities are functions of two or more variables and du, dv, etc., are their total differentials.

Take, for example, the differential of uv. By (93c) the result is

$$d\,(uv) = \frac{\partial}{\partial u}\,(uv)\,du + \frac{\partial}{\partial v}\,(uv)\,dv = v\,du + u\,dv,$$

which is the formula IV of Art. 17.

Example. $u = ye^x + ze^y$.

Differentiating term by term, we get

$$du = ye^x\,dx + e^x\,dy + ze^y\,dy + e^y\,dz.$$

We obtain the same result by using (93d); for that formula gives

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy + \frac{\partial u}{\partial z}\,dz = ye^x\,dx + (e^x + ze^y)\,dy + e^y\,dz.$$

95. Partial Derivatives as Ratios of Differentials. — The equation

$$d_xu = \frac{\partial u}{\partial x}\,dx$$

shows that the partial derivative $\frac{\partial u}{\partial x}$ is the ratio of two differentials d_xu and dx. Now d_xu is the value of du when the same quantities are kept constant that are constant in the calculation of $\frac{\partial u}{\partial x}$. Therefore, *the partial derivative* $\frac{\partial u}{\partial x}$ *is the* *value to which* $\frac{du}{dx}$ *reduces when du and dx are determined with the same quantities constant that are constant in the calculation of* $\frac{\partial u}{\partial x}$.

Example. Given $u = x^2 + y^2 + z^2$, $v = xyz$, find $\left(\frac{\partial u}{\partial x}\right)_{v,z}$.

Differentiating the two equations with v and z constant, we get

$$du = 2\,x\,dx + 2\,y\,dy, \qquad 0 = yz\,dx + xz\,dy.$$

Eliminating dy,

$$du = 2\,x\,dx - 2\,\frac{y^2}{x}\,dx = 2\left(\frac{x^2 - y^2}{x}\right)dx.$$

Under the given conditions the ratio of du to dx is then

$$\frac{du}{dx} = \frac{2\,(x^2 - y^2)}{x}.$$

Since v and z were kept constant, this ratio represents $\left(\frac{\partial u}{\partial x}\right)_{v,z}$; that is,

$$\left(\frac{\partial u}{\partial x}\right)_{v,z} = \frac{2\,(x^2 - y^2)}{x}.$$

EXERCISES

1. One side of a right triangle increases from 5 to 5.2 while the other decreases from 12 to 11.75. Find the increment of the hypotenuse and its principal part.

2. A closed box, 12 in. long, 8 in. wide, and 6 in. deep, is made of material $\frac{1}{4}$ inch thick. Find approximately the volume of material used.

3. Two sides and the included angle of a triangle are $b = 20$, $c = 30$, and $A = 45°$. By using the formula

$$a^2 = b^2 + c^2 - 2\,bc\cos A,$$

find approximately the change in a when b increases 1 unit, c decreases $\frac{1}{2}$ unit, and A increases 1 degree.

4. The period of a simple pendulum is

$$T = 2\,\pi\sqrt{\frac{l}{g}}.$$

Find the error in T due to small errors in l and g.

5. If g is computed by the formula,

$$s = \tfrac{1}{2} gt^2,$$

find the error in g due to small errors in s and t.

6. The area of a triangle is determined by the formula

$$K = \tfrac{1}{2} ab \sin C.$$

Find the error in K due to small errors in a, b, C.

Find the total differentials of the following functions:

7. xy^2z^3.

8. $xy \sin (x + y)$.

9. $\dfrac{x}{y} + \dfrac{y}{z} + \dfrac{z}{x}$.

10. $\tan^{-1} \dfrac{y}{x} + \tan^{-1} \dfrac{x}{y}$.

11. The pressure, volume, and temperature of a perfect gas are connected by the equation $pv = kt$, k being constant. Find dp in terms of dv and dt.

12. If x, y are rectangular and r, θ polar coördinates of the same point, show that

$$x\,dy - y\,dx = r^2\,d\theta, \quad dx^2 + dy^2 = dr^2 + r^2\,d\theta^2.$$

13. If $x = u - v$, $y = u^2 + v^2$, find $\left(\dfrac{\partial x}{\partial v}\right)_y$.

14. If $u = xy + yz + zx$, $x^2 + z^2 = 2\,yz$, find $\left(\dfrac{\partial u}{\partial z}\right)_y$.

15. If $yz = ux + v^2$, $vx = uy + z^2$, find $\left(\dfrac{\partial v}{\partial z}\right)_{u,\,x}$.

16. A variable triangle with sides a, b, c and opposite angles A, B, C is inscribed in a fixed circle. Show that

$$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0.$$

96. Derivative of a Function of Several Variables. — Let $u = f(x, y)$ and let x and y be functions of two variables s and t. When t changes to $t + \Delta t$, x and y will change to $x + \Delta x$ and $y + \Delta y$. The resulting increment in u will be

$$\Delta u = \frac{\partial u}{\partial x}\,\Delta x + \frac{\partial u}{\partial y}\,\Delta y + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y.$$

Consequently,

$$\frac{\Delta u}{\Delta t} = \frac{\partial u}{\partial x}\,\frac{\Delta x}{\Delta t} + \frac{\partial u}{\partial y}\,\frac{\Delta y}{\Delta t} + \epsilon_1\,\frac{\Delta x}{\Delta t} + \epsilon_2\,\frac{\Delta y}{\Delta t}.$$

As Δt approaches zero, Δx and Δy will approach zero and so

ϵ_1 and ϵ_2 will approach zero. Taking the limit of both sides,

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}. \tag{96a}$$

If x or y is a function of t only, the partial derivative $\dfrac{\partial x}{\partial t}$ or $\dfrac{\partial y}{\partial t}$ is replaced by a total derivative $\dfrac{dx}{dt}$ or $\dfrac{dy}{dt}$. If both x and y are functions of t, u is a function of t with total derivative

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}. \tag{96b}$$

Likewise, if u is a function of three variables x, y, z, that depend on t,

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}. \tag{96c}$$

As before, if a variable is a function of t only, its partial derivative is replaced by a total one. Similar results hold for any number of variables.

The term

$$\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}$$

is the result of differentiating u with respect to t, leaving all the variables in u except x constant. Equations (96a) and (96c) express that *if u is a function of several variable quantities, $\dfrac{\partial u}{\partial t}$ can be obtained by differentiating with respect to t as if only one of those quantities were variable at a time and adding the results.*

Example 1. Given $y = x^x$, find $\dfrac{dy}{dx}$.

The function x^x can be considered a function of two variables, the lower x and the upper x. If the upper x is held constant and the lower allowed to vary, the derivative (as in case of x^n) is

$$x \cdot x^{x-1} = x^x.$$

If the lower x is held constant while the upper varies, the derivative (as in case of a^x) is

$$x^x \ln x.$$

The actual derivative of y is then the sum

$$\frac{dy}{dx} = x^x + x^x \ln x.$$

Ex. 2. Given $u = f(x, y, z)$, y and z being functions of x; find $\frac{du}{dx}\cdot$

By equation (96c) the result is

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} + \frac{\partial u}{\partial z}\frac{dz}{dx}\cdot$$

In this equation there are two derivatives of u with respect to x. If y and z are replaced by their values in terms of x, u will be a function of x only. The derivative of that function is $\frac{du}{dx}\cdot$ If y and z are replaced by constants, u will be a second function of x. Its derivative is $\frac{\partial u}{\partial x}\cdot$

Ex. 3. Given $u = f(x, y, z)$, z being a function of x and y. Find the partial derivative of u with respect to x.

It is understood that y is to be constant in this partial differentiation. Equation (96c) then gives

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\cdot$$

In this equation appear two partial derivatives of u with respect to x. If z is replaced by its value in terms of x and y, u will be expressed as a function of x and y only. Its partial derivative is the one on the left side of the equation. If z is kept constant, u is again a function of x and y. Its partial derivative appears on the right side of the equation. We must not of course use the same symbol for both of these derivatives. A way to avoid the confusion is to use the letter f instead of u on the right side of the equation. It then becomes

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x}.$$

It is understood that $f(x, y, z)$ is a definite function of x, y, z and that $\frac{\partial f}{\partial x}$ is the derivative obtained with all the variables but x constant.

97. Change of Variable. — If u is a function of x and y we have said that the equation

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$$

is true whether x and y are the independent variables or not. To show this let s and t be the independent variables and x and y functions of them. Then, by definition,

$$du = \frac{\partial u}{\partial s}\,ds + \frac{\partial u}{\partial t}\,dt.$$

Since u is a function of x and y which are functions of s and t, by equation (96a),

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}, \qquad \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}.$$

Consequently,

$$\begin{aligned} du &= \left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}\right)ds + \left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}\right)dt \\ &= \frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial s}\,ds + \frac{\partial x}{\partial t}\,dt\right) + \frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial s}\,ds + \frac{\partial y}{\partial t}\,dt\right) = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy, \end{aligned}$$

which was to be proved.

A similar proof can be given in case of three or more variables.

98. Implicit Functions. — If two or more variables are connected by an equation, a differential relation can be obtained by equating the total differentials of the two sides of the equation.

Example 1. $f(x, y) = 0$.

In this case

$$d \cdot f(x, y) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = d \cdot 0 = 0.$$

Consequently,

$$\frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}}.$$

Ex. 2. $f(x, y, z) = 0$.

Differentiation gives

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = 0.$$

If z is considered a function of x and y, its partial derivative with respect to x is found by keeping y constant. Then $dy = 0$ and

$$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial z}}.$$

Similarly, if x is constant, $dx = 0$ and

$$\frac{\partial z}{\partial y} = -\frac{\dfrac{\partial f}{\partial y}}{\dfrac{\partial f}{\partial z}}.$$

Ex. 3. $f_1(x, y, z) = 0$, $f_2(x, y, z) = 0$.

We have two differential relations

$$\frac{\partial f_1}{\partial x} dx + \frac{\partial f_1}{\partial y} dy + \frac{\partial f_1}{\partial z} dz = 0,$$

$$\frac{\partial f_2}{\partial x} dx + \frac{\partial f_2}{\partial y} dy + \frac{\partial f_2}{\partial z} dz = 0.$$

We could eliminate y from the two equations $f_1 = 0$, $f_2 = 0$. We should then obtain z as a function of x. The total derivative of this function is found by eliminating dy and solving for the ratio $\frac{dz}{dx}$. The result is

$$\frac{dz}{dx} = \frac{\frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial y}}{\frac{\partial f_1}{\partial z}\frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial z}}.$$

99. Directional Derivative. — Let $u = f(x, y)$. At each point $P(x, y)$ in the xy-plane, u has a definite value. If we move away from P in any definite direction PQ, x and y will

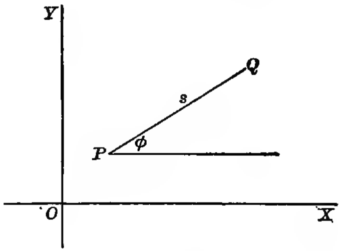

FIG. 99.

be functions of the distance moved. The derivative of u with respect to s is

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds} = \frac{\partial u}{\partial x}\cos\phi + \frac{\partial u}{\partial y}\sin\phi.$$

This is called the derivative of u in the direction PQ. The partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are special values of $\frac{\partial u}{\partial s}$ which result when PQ is drawn in the direction of OX or OY.

Similarly, if $u = f(x, y, z)$,

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds} + \frac{\partial u}{\partial z}\frac{dz}{ds} = \frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\cos\beta + \frac{\partial u}{\partial z}\cos\gamma$$

is the rate of change of u with respect to s as we move along a line with direction cosines $\cos\alpha$, $\cos\beta$, $\cos\gamma$. The partial derivatives of u are the values to which $\frac{\partial u}{\partial s}$ reduces when s is measured in the direction of a coördinate axis.

Example. Find the derivative of $x^2 + y^2$ in the direction $\phi = 45°$ at the point (1, 2).

The result is

$$\frac{\partial}{\partial s}(x^2 + y^2) = 2\,x\frac{\partial x}{\partial s} + 2\,y\frac{\partial y}{\partial s} = 2\,x\cos\phi + 2\,y\sin\phi$$
$$= 2\cdot\frac{1}{\sqrt{2}} + 4\cdot\frac{1}{\sqrt{2}} = 3\sqrt{2}.$$

100. Exact Differentials. — If P and Q are functions of two independent variables x and y,

$$P\,dx + Q\,dy$$

may or may not be the total differential of a function u of x and y. If it is the total differential of such a function,

$$P\,dx + Q\,dy = du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

Since dx and dy are arbitrary, this requires

$$P = \frac{\partial u}{\partial x}, \qquad Q = \frac{\partial u}{\partial y}.$$

Consequently,

$$\frac{\partial P}{\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}, \qquad \frac{\partial Q}{\partial x} = \frac{\partial^2 u}{\partial x\,\partial y}.$$

Since the two second derivatives of u with respect to x and y are equal,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \tag{100a}$$

An expression $P\,dx + Q\,dy$ is called an *exact differential* if it is the total differential of a function of x and y. We have just shown that (100a) must then be satisfied. Conversely, it can be shown that if this equation is satisfied $P\,dx + Q\,dy$ is an exact differential.*

* See Wilson, *Advanced Calculus*, § 92.

Similarly, if

$$P\,dx + Q\,dy + R\,dz$$

is the differential of a function u of x, y, z,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \qquad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}, \tag{100b}$$

and conversely.

Example 1. Show that

$$(x^2 + 2\,xy)\,dx + (x^2 + y^2)\,dy$$

is an exact differential.

In this case

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(x^2 + 2\,xy) = 2\,x, \qquad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2) = 2\,x.$$

The two partial derivatives being equal, the expression is exact.

Ex. 2. In thermodynamics it is shown that

$$dU = T\,dS - p\,dv,$$

U being the internal energy, T the absolute temperature, S the entropy, p the pressure, and v the volume of a homogeneous substance. Any two of these five quantities can be assigned independently and the others are then determined. Show that

$$\left(\frac{\partial T}{\partial p}\right)_S = \left(\frac{\partial v}{\partial S}\right)_p.$$

The result to be proved expresses that

$$T\,dS + v\,dp$$

is an exact differential. That such is the case is shown by replacing $T\,dS$ by its value $dU + p\,dv$. We thus get

$$T\,dS + v\,dp = dU + p\,dv + v\,dp = d\,(U + pv).$$

EXERCISES

1. If $u = f(x, y)$, $y = \phi(x)$, find $\dfrac{du}{dx}$.

2. If $u = f(x, y, z)$, $z = \phi(x)$, find $\left(\dfrac{\partial u}{\partial x}\right)_y$.

3. If $u = f(x, y, z)$, $z = \phi(x, y)$, $y = \psi(x)$, find $\frac{du}{dx}$.

4. If $u = f(x, y)$, $y = \phi(x, r)$, $r = \psi(x, s)$, find $\left(\frac{\partial u}{\partial x}\right)_y$, $\left(\frac{\partial u}{\partial x}\right)_r$, and $\left(\frac{\partial u}{\partial x}\right)_s$.

5. If $f(x, y, z) = 0$, $z = F(x, y)$, find $\frac{dz}{dx}$.

6. If $F(x, y, z) = 0$, show that

$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x} = -1.$$

7. If $u = xf(z)$, $z = \frac{y}{x}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$.

8. If $u = f(r, s)$, $r = x + at$, $s = y + bt$, show that $\frac{\partial u}{\partial t} = a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y}$.

9. If $z = f(x + ay)$, show that $\frac{\partial z}{\partial y} = a\frac{\partial z}{\partial x}$.

10. If $u = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$, show that

$$\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2.$$

11. The position of a pair of rectangular axes moving in a plane is determined by the coördinates h, k of the moving origin and the angle ϕ between the moving x-axis and a fixed one. A variable point P has coördinates x', y' with respect to the moving axes and x, y with respect to the fixed ones. Then

$$x = f(x', y', h, k, \phi), \qquad y = F(x', y', h, k, \phi).$$

Find the velocity of P. Show that it is the sum of two parts, one representing the velocity the point would have if it were rigidly connected with the moving axes, the other representing its velocity with respect to those axes conceived as fixed.

12. Find the directional derivatives of the rectangular coördinates x, y and the polar coördinates r, θ of a point in a plane. Show that they are identical with the derivatives with respect to s given in Arts. 54 and 59.

13. Find the derivative of $x^2 - y^2$ in the direction $\phi = 30°$ at the point (3, 4).

14. At a distance r in space the potential due to an electric charge e is $V = \frac{e}{r}$. Find its directional derivative.

15. Show that the derivative of xy along the normal at any point of the curve $x^2 - y^2 = a^2$ is zero.

16. Given $u = f(x, y)$, show that

$$\left(\frac{\partial u}{\partial s_1}\right)^2 + \left(\frac{\partial u}{\partial s_2}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2,$$

if s_1 and s_2 are measured along perpendicular directions.

Determine which of the following expressions are exact differentials:

17. $y\,dx - x\,dy$.

18. $(2x + y)\,dx + (x - 2y)\,dy$.

19. $e^x dx + e^y dy + (x + y)\,e^z dz$.

20. $yz\,dx - xz\,dy + y^2 dz$.

21. Under the conditions of Ex. 2, page 131, show that

$$\left(\frac{\partial v}{\partial T}\right)_p = -\left(\frac{\partial S}{\partial p}\right)_T, \qquad \left(\frac{\partial p}{\partial T}\right)_v = \left(\frac{\partial S}{\partial v}\right)_T.$$

22. In case of a perfect gas, $pv = kT$. Using this and the equation

$$dU = T\,dS - p\,dv,$$

show that

$$\frac{\partial U}{\partial p} = 0.$$

Since U is always a function of p and T, this last equation expresses that U is a function of T only.

101. Direction of the Normal at a Point of a Surface. — Let the equation of a surface be

$$F(x, y, z) = 0.$$

Differentiation gives

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz = 0. \tag{101a}$$

Let PN be the line through $P\ (x, y, z)$ with direction cosines proportional to

$$\frac{\partial F}{\partial x} : \frac{\partial F}{\partial y} : \frac{\partial F}{\partial z}.$$

If P moves along a curve on the surface, the direction cosines of its tangent PT are proportional to

$$dx : dy : dz.$$

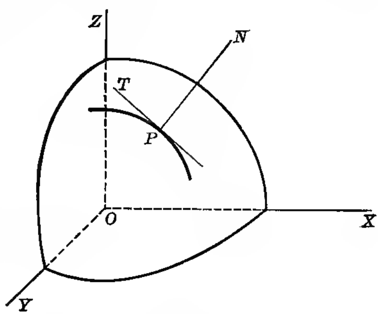

FIG. 101.

Equation (101a) expresses that PN and PT are perpendicular to each other (Art. 61). Consequently PN is perpendicular to all the tangent lines through P. This is expressed by saying PN is the *normal* to the surface at P. We conclude that *the normal to the surface* $F(x, y, z) = 0$ *at* $P(x, y, z)$ *has direction cosines proportional to*

$$\frac{\partial F}{\partial x} : \frac{\partial F}{\partial y} : \frac{\partial F}{\partial z}. \tag{101}$$

102. Equations of the Normal at $P_1(x_1, y_1, z_1)$.—Let A, B, C be proportional to the direction cosines of the normal to a surface at $P_1(x_1, y_1, z_1)$. The equations of the normal are (Art. 63)

$$\frac{x - x_1}{A} = \frac{y - y_1}{B} = \frac{z - z_1}{C}. \tag{102}$$

103. Equation of the Tangent Plane at $P_1(x_1, y_1, z_1)$. — All the tangent lines at P_1 on the surface are perpendicular

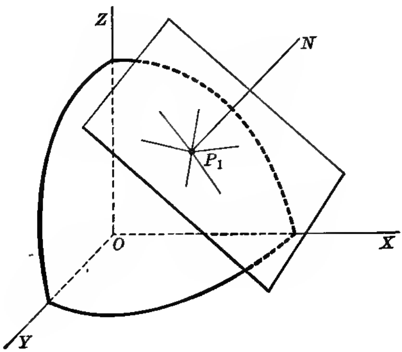

FIG. 103.

to the normal at that point. All these lines therefore lie in a plane perpendicular to the normal, called the *tangent plane* at P_1.

It is shown in analytical geometry that if A, B, C are proportional to the direction cosines of the normal to a plane passing through (x_1, y_1, z_1), the equation of the plane is

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0.^* \tag{103}$$

* See Phillips, *Analytic Geometry*, Art. 68.

If A, B, C are proportional to the direction cosines of the normal to a surface at P_1, this is then the equation of the tangent plane at P_1.

Example. Find the equations of the normal line and tangent plane at the point $(1, -1, 2)$ of the ellipsoid

$$x^2 + 2y^2 + 3z^2 = 3x + 12.$$

The equation given is equivalent to

$$x^2 + 2y^2 + 3z^2 - 3x - 12 = 0.$$

The direction cosines of its normal are proportional to the partial derivatives

$$2x - 3 : 4y : 6z.$$

At the point $(1, -1, 2)$, these are proportional to

$$A : B : C = -1 : -4 : 12 = 1 : 4 : -12.$$

The equations of the normal are

$$\frac{x-1}{1} = \frac{y+1}{4} = \frac{z-2}{-12}.$$

The equation of the tangent plane is

$$x - 1 + 4(y+1) - 12(z-2) = 0.$$

EXERCISES

Find the equations of the normal and tangent plane to each of the following surfaces at the point indicated:

1. Sphere, $x^2 + y^2 + z^2 = 9$, at $(1, 2, 2)$.
2. Cylinder, $x^2 + xy + y^2 = 7$, at $(2, -3, 3)$.
3. Cone, $z^2 = x^2 + y^2$, at $(3, 4, 5)$.
4. Hyperbolic paraboloid, $xy = 3z - 4$, at $(5, 1, 3)$.
5. Elliptic paraboloid, $x = 2y^2 + 3z^2$, at $(5, 1, 1)$.
6. Find the locus of points on the cylinder

$$(x+z)^2 + (y-z)^2 = 4$$

where the normal is parallel to the xy-plane.

7. Show that the normal at any point $P(x, y, z)$ of the surface $y^2 + z^2 = 4x$ makes equal angles with the x-axis and the line joining P and $A(1, 0, 0)$.

8. Show that the normal to the spheroid

$$\frac{x^2 + z^2}{9} + \frac{y^2}{25} = 1$$

at $P(x, y, z)$ determines equal angles with the lines joining P with $A'(0, -4, 0)$ and $A(0, 4, 0)$.

104. Maxima and Minima of Functions of Several Variables. — A *maximum* value of a function u is a value greater than any given by neighboring values of the variables. In passing from a maximum to a neighboring value, the function decreases, that is

$$\Delta u < 0. \tag{104a}$$

A *minimum* value is a value less than any given by neighboring values of the variables. In passing from a minimum to a neighboring value

$$\Delta u > 0. \tag{104b}$$

If the condition (104a) or (104b) is satisfied for all small changes of the variables, it must be satisfied when a single variable changes. If then all the independent variables but x are kept constant, u must be a maximum or minimum in x. If $\frac{\partial u}{\partial x}$ is continuous, by Art. 31,

$$\frac{\partial u}{\partial x} = 0. \tag{104c}$$

Therefore, *if the first partial derivatives of u with respect to the independent variables are continuous, those derivatives must be zero when u is a maximum or minimum.*

When the partial derivatives are zero, the total differential is zero. For example, if x and y are the independent variables,

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy = 0 \cdot dx + 0 \cdot dy = 0. \tag{104d}$$

Therefore, *if the first partial derivatives are continuous, the total differential of u is zero when u is either a maximum or a minimum.*

To find the maximum and minimum values of a function, we equate its differential or the partial derivatives with respect to the *independent* variables to zero and solve the resulting equations. It is usually possible to decide from the problem whether a value thus found is a maximum, minimum, or neither.

Example 1. Show that the maximum rectangular parallelopiped with a given area of surface is a cube.

Let x, y, z be the edges of the parallelopiped. If V is the volume and A the area of its surface

$$V = xyz, \qquad A = 2\,xy + 2\,xz + 2\,yz.$$

Two of the variables x, y, z are independent. Let them be x, y. Then

$$z = \frac{A - 2\,xy}{2\,(x + y)}.$$

Therefore

$$V = \frac{xy\,(A - 2\,xy)}{2\,(x + y)},$$

$$\frac{\partial V}{\partial x} = \frac{y^2}{2}\left[\frac{A - 2\,x^2 - 4\,xy}{(x + y)^2}\right] = 0,$$

$$\frac{\partial V}{\partial y} = \frac{x^2}{2}\left[\frac{A - 2\,y^2 - 4\,xy}{(x + y)^2}\right] = 0.$$

The values $x = 0$, $y = 0$ cannot give maxima. Hence

$$A - 2\,x^2 - 4\,xy = 0, \qquad A - 2\,y^2 - 4\,xy = 0.$$

Solving these equations simultaneously with

$$A = 2\,xy + 2\,xz + 2\,yz,$$

we get

$$x = y = z = \sqrt{\frac{A}{6}}.$$

We know there is a maximum. Since the equations give only one solution it must be the maximum.

Ex. 2. Find the point in the plane

$$x + 2\,y + 3\,z = 14$$

nearest to the origin.

The distance from any point (x, y, z) of the plane to the origin is

$$D = \sqrt{x^2 + y^2 + z^2}.$$

If this is a minimum

$$d \cdot D = \frac{x\,dx + y\,dy + z\,dz}{\sqrt{x^2 + y^2 + z^2}} = 0,$$

that is,

$$x\,dx + y\,dy + z\,dz = 0. \tag{104e}$$

From the equation of the plane we get

$$dx + 2\,dy + 3\,dz = 0. \tag{104f}$$

The only equation connecting x, y, z is that of the plane. Consequently, dx, dy, dz can have any values satisfying this last equation. If x, y, z are so chosen that D is a minimum (104e) must be satisfied by all of these values. If two linear equations have the same solutions, one is a multiple of the other. Corresponding coefficients are proportional. The coefficients of dx, dy, dz in (104e) are x, y, z. Those in (104f) are 1, 2, 3. Hence

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3}.$$

Solving these simultaneously with the equation of the plane, we get $x = 1$, $y = 2$, $z = 3$. There is a minimum. Since we get only one solution, it is the minimum.

EXERCISES

1. An open rectangular box is to have a given capacity. Find the dimensions of the box requiring the least material.

2. A tent having the form of a cylinder surmounted by a cone is to contain a given volume. Find its dimensions if the canvas required is a minimum.

3. When an electric current of strength I flows through a wire of resistance R the heat produced is proportional to I^2R. Two terminals are connected by three wires of resistances R_1, R_2, R_3 respectively. A given current flowing between the terminals will divide between the wires in such a way that the heat produced is a minimum. Show that the currents I_1, I_2, I_3 in the three wires will satisfy the equations

$$I_1R_1 = I_2R_2 = I_3R_3.$$

4. A particle attracted toward each of three points A, B, C with a force proportional to the distance will be in equilibrium when the sum of the squares of the distances from the points is least. Find the position of equilibrium.

5. Show that the triangle of greatest area with a given perimeter is equilateral.

6. Two adjacent sides of a room are plane mirrors. A ray of light starting at P strikes one of the mirrors at Q, is reflected to a point R on the second mirror, and is there reflected to S. If P and S are in the same horizontal plane find the positions of Q and R so that the path $PQRS$ may be as short as possible.

7. A table has four legs attached to the top at the corners A_1, A_2, A_3, A_4 of a square. A weight W placed upon the table at a point of the diagonal A_1A_3, two-thirds of the way from A_1 to A_3, will cause the legs to shorten the amounts s_1, s_2, s_3, s_4, while the weight itself sinks a distance h. The increase in potential energy due to the contraction of a leg is ks^2, where k is constant and s the contraction. The decrease in potential energy due to the sinking of the weight is Wh. The whole system will settle to a position such that the potential energy is a minimum. Assuming that the top of the table remains plane, find the ratios of s_1, s_2, s_3, s_4.

SUPPLEMENTARY EXERCISES

CHAPTER III

Find the differentials of the following functions:

1. $\dfrac{\sqrt{ax^2+b}}{bx}$.

2. $\dfrac{x}{b\sqrt{ax^2+b}}$.

3. $\dfrac{2\sqrt{ax^2+bx}}{bx}$.

4. $\dfrac{2ax+b}{\sqrt{ax^2+bx+c}}$.

5. $\dfrac{(ax+b)^{n+2}}{a^2(n+2)} - \dfrac{b(ax+b)^{n+1}}{a^2(n+1)}$.

6. $x(a^2+x^2)\sqrt{a^2-x^2}$.

7. $\dfrac{(2x+1)(2x+7)^2}{(2x+5)^3}$.

8. $\dfrac{(x+2)^6(x+4)^2}{(x+1)^2(x+3)^6}$.

9. $\dfrac{(2x^2-1)\sqrt{x^2+1}}{x^3}$.

10. $x(x^n+n)^{\frac{n-1}{n}}$.

Find $\dfrac{dy}{dx}$ in each of the following cases:

11. $2x^2 - 4xy + 3y^2 = 6x - 4y + 18$.
12. $x^3 + 3x^2y = y^3$.
13. $x = 3y^2 + 2y^3$.
14. $(x^2+y^2)^2 = 2a^2(x^2-y^2)$.
15. $x = t + \dfrac{1}{t-1}, \quad y = 2t - \dfrac{1}{(t-1)^2}$.
16. $x = \dfrac{t}{\sqrt{1+t^2}}, \quad y = \dfrac{1}{\sqrt{1-t^2}}$.
17. $x = t(t^2+a^2)^{\frac{1}{2}} \quad y = t(t^2+a^2)^{\frac{3}{2}}$.
18. $x = z^2 + 2z, \quad z = y^2 + 2y$.
19. $x^2 + z^2 = a^2, \quad yz = b^2$.

20. The volume elasticity of a fluid is $e = -v\dfrac{dp}{dv}$. If a gas expands according to Boyle's law, $pv =$ constant, show that $e = p$.

21. When a gas expands without receiving or giving out heat, the pressure, volume, and temperature satisfy the equations

$$pv = RT, \quad pv^n = C,$$

R, n, and C being constants. Find $\dfrac{dp}{dT}$ and $\dfrac{dv}{dT}$.

22. If v is the volume of a spherical segment of altitude h, show that $\frac{dv}{dh}$ is equal to the area of the circle forming the plane face of the segment.

23. If a polynomial equation

$$f(x) = 0$$

has two roots equal to r, $f(x)$ has $(x - r)^2$ as a factor, that is,

$$f(x) = (x - r)^2 f_1(x),$$

where $f_1(x)$ is a polynomial in x. Hence show that r is a root of

$$f'(x) = 0,$$

where $f'(x)$ is the derivative of $f(x)$.

Show by the method of Ex. 23 that each of the following equations has a double root and find it:

24. $x^3 - 3x^2 + 4 = 0.$

25. $x^3 - x^2 - 5x - 3 = 0.$

26. $4x^3 - 8x^2 - 3x + 9 = 0.$

27. $4x^4 - 12x^3 + x^2 + 12x + 4 = 0.$

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in each of the following cases.

28. $y = x\sqrt{a^2 - x^2}.$

29. $y = \frac{x^2}{(x+1)^2}.$

30. $xy = a^2.$

31. $ax + by + c = 0.$

32. $x = 2 + 3t,\ y = 4 - 5t.$

33. $x = \frac{t}{t+1},\quad y = \frac{t^2}{t+1}.$

34. If $y = x^2$, find $\frac{d^2y}{dx^2}$ and $\frac{d^2x}{dy^2}$.

35. Given $x^2 - y^2 = 1$, verify that

$$\frac{d^2y}{dx^2} = -\frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)^3.$$

36. If n is a positive integer, show that

$$\frac{d^n}{dx^n}x^n = \text{constant}.$$

37. If u and v are functions of x, show that

$$\frac{d^4}{dx^4}(uv) = \frac{d^4u}{dx^4}\cdot v + 4\frac{d^3u}{dx^3}\cdot\frac{dv}{dx} + 6\frac{d^2u}{dx^2}\cdot\frac{d^2v}{dx^2} + 4\frac{du}{dx}\cdot\frac{d^3v}{dx^3} + u\frac{d^4v}{dx^4}.$$

Compare this with the binomial expansion for $(u + v)^4$.

38. If $f(x) = (x - r)^3 f_1(x)$, where $f_1(x)$ is a polynomial, show that

$$f'(r) = f''(r) = 0.$$

CHAPTER IV

39. A particle moves along a straight line the distance

$$s = 4t^3 - 21t^2 + 36t + 1$$

feet in t seconds. Find its velocity and acceleration. When is the particle moving forward? When backward? When is the velocity increasing? When decreasing?

40. Two trains start from different points and move along the same track in the same direction. If the train in front moves a distance $6t^3$ in t hours and the rear one $12t^2$, how fast will they be approaching or separating at the end of one hour? At the end of two hours? When will they be closest together?

41. If $s = \sqrt{t}$, show that the acceleration is negative and proportional to the cube of the velocity.

42. The velocity of a particle moving along a straight line is

$$v = 2t^2 - 3t.$$

Find its acceleration when $t = 2$.

43. If $v^2 = \frac{k}{s}$, where k is constant, find the acceleration.

44. Two wheels, diameters 3 and 5 ft., are connected by a belt. What is the ratio of their angular velocities and which is greater? What is the ratio of their angular accelerations?

45. Find the angular velocity of the earth about its axis assuming that there are $365\frac{1}{4}$ days in a year.

46. A wheel rolls down an inclined plane, its center moving the distance $s = 5t^2$ in t seconds. Show that the acceleration of the wheel about its axis is constant.

47. An amount of money is drawing interest at 6 per cent. If the interest is immediately added to the principal, what is the rate of change of the principal?

48. If water flows from a conical funnel at a rate proportional to the square root of the depth, at what rate does the depth change?

49. A kite is 300 ft. high and there are 300 ft. of cord out. If the kite moves horizontally at the rate of 5 miles an hour directly away from the person flying it, how fast is the cord being paid out?

50. A particle moves along the parabola

$$100y = 16x^2$$

in such a way that its abscissa changes at the rate of 10 ft./sec. Find the velocity and acceleration of its projection on the y-axis.

51. The side of an equilateral triangle is increasing at the rate of 10 ft. per minute and its area at the rate of 100 sq. ft. per minute. How large is the triangle?

CHAPTER V

52. The velocity of waves of length λ in deep water is proportional to

$$\sqrt{\frac{\lambda}{a} + \frac{a}{\lambda}}$$

when a is a constant. Show that the velocity is a minimum when $\lambda = a$.

53. The sum of the surfaces of a sphere and cube is given. Show that the sum of the volumes is least when the diameter of the sphere equals the edge of the cube.

54. A box is to be made out of a piece of cardboard, 6 inches square, by cutting equal squares from the corners and turning up the sides. Find the dimensions of the largest box that can be made in this way.

55. A gutter of trapezoidal section is made by joining 3 pieces of material each 4 inches wide, the middle one being horizontal. How wide should the gutter be at the top to have the maximum capacity?

56. A gutter of rectangular section is to be made by bending into shape a strip of copper. Show that the capacity of the gutter will be greatest if its width is twice its depth.

57. If the top and bottom margins of a printed page are each of width a, the side margins of width b, and the text covers an area c, what should be the dimensions of the page to use the least paper?

58. Find the dimensions of the largest cone that can be inscribed in a sphere of radius a.

59. Find the dimensions of the smallest cone that can contain a sphere of radius a.

60. To reduce the friction of a liquid against the walls of a channel, the channel should be so designed that the area of wetted surface is as small as possible. Show that the best form for an open rectangular channel with given cross section is that in which the width equals twice the depth.

61. Find the dimensions of the best trapezoidal channel, the banks making an angle θ with the vertical.

62. Find the least area of canvas that can be used to make a conical tent of 1000 cu. ft. capacity.

63. Find the maximum capacity of a conical tent made of 100 sq. ft. of canvas.

64. Find the height of a light above the center of a table of radius a, so as best to illuminate a point at the edge of the table; assuming that the illumination varies inversely as the square of the distance from the light and directly as the sine of the angle between the rays and the surface of the table.

65. A weight of 100 lbs., hanging 2 ft. from one end of a lever, is to be raised by an upward force applied at the other end. If the lever weighs 3 lbs. to the foot, find its length so that the force may be a minimum.

66. A vertical telegraph pole at a bend in the line is to be supported from tipping over by a stay 40 ft. long fastened to the pole and to a stake in the ground. How far from the pole should the stake be driven to make the tension in the stay as small as possible?

67. The lower corner of a leaf of a book is folded over so as just to reach the inner edge of the page. If the width of the page is 6 inches, find the width of the part folded over when the length of the crease is a minimum.

68. If the cost of fuel for running a train is proportional to the square of the speed and \$10 per hour for a speed of 12 mi./hr., and the fixed charges on \$90 per hour, find the most economical speed.

69. If the cost of fuel for running a steamboat is proportional to the cube of the speed and \$10 per hour for a speed of 10 mi./hr., and the fixed charges are \$14 per hour, find the most economical speed against a current of 2 mi./hr.

CHAPTER VI

Differentiate the following functions:

70. $\dfrac{\sin x}{x}$.

71. $\dfrac{\sin\theta}{1-\cos\theta}$.

72. $\dfrac{1+\cos\theta}{\sin\theta}$.

73. $\sin ax \cos ax$.

74. $\cot\dfrac{\theta}{2} - \csc\dfrac{\theta}{2}$.

75. $\tan 2x - \cot 2x$.

76. $\sec^2 x - \tan^2 x$.

77. $\sin^3\dfrac{3}{x}\sec\dfrac{x}{3}$.

78. $\tan\dfrac{x}{1-x}$.

79. $\dfrac{2\tan x}{1-\tan^2 x}$.

80. $5\sec^7\theta - 7\sec^5\theta$.

81. $\sec x \csc x - 2\cot x$.

Differentiate both sides of each of the following equations and show that the resulting derivatives are equal.

82. $\sec^2 x + \csc^2 x = \sec^2 x \csc^2 x$.

83. $\sin 2x = 2\sin x \cos x$.

84. $\sin 3x = 3\sin x - 4\sin^3 x$.

85. $\sin(x+a) = \sin x \cos a + \cos x \sin a$.

86. $\sec^2 x = 1 + \tan^2 x$.

87. $\sin x + \sin a = 2\sin\frac{1}{2}(x+a)\cos\frac{1}{2}(x-a)$.

88. $\cos a - \cos x = 2 \sin \frac{1}{2}(x + a) \sin \frac{1}{2}(x - a)$.

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in each of the following cases:

89. $x = a \cos^2 \theta, \quad y = a \sin^2 \theta$.
90. $x = a \cos^5 \theta, \quad y = a \sin^5 \theta$.
91. $x = \tan \theta - \theta, \quad y = \cos \theta$.
92. $x = \sec^2 \theta, \quad y = \tan^2 \theta$.
93. $x = \sec \theta, \quad y = \tan \theta$.
94. $x = \csc \theta - \cot \theta, \quad y = \csc \theta + \cot \theta$.

Differentiate the following functions:

95. $\sin^{-1} \sqrt{\frac{x}{2}}$.
96. $\cos^{-1} \left(\frac{1}{x}\right)$.
97. $\tan^{-1} \left(\frac{1 - 2x}{3}\right)$.
98. $\frac{2}{\sqrt{3}} \tan^{-1} \frac{2x + 1}{\sqrt{3}}$.
99. $\cos^{-1} \frac{x}{\sqrt{x^2 + 1}}$.
100. $\csc^{-1} \frac{\sqrt{5}}{2x - 1}$.
101. $\sec^{-1} \frac{1}{2}\left(x + \frac{1}{x}\right)$.
102. $a \csc^{-1} \frac{a}{x} + \sqrt{a^2 - x^2}$.
103. $\frac{x}{1 + x^2} - \cot^{-1} x$.
104. $\sqrt{1 - x} \sin^{-1} x - \sqrt{x}$.
105. $\sec^{-1} \frac{x + 1}{x - 1} + \sin^{-1} \frac{x - 1}{x + 1}$.
106. $\sin^{-1} \frac{a + b \cos x}{b + a \cos x}$.
107. $\frac{1}{2} \cos^{-1} x + \frac{b}{2} \sqrt{1 - x^2}$.
108. $\sqrt{x^2 - a^2} - a \sec^{-1} \frac{x}{a}$.

109. $e^{\sqrt{x}}$.
110. $\sqrt{e^x}$.
111. $(\sqrt{e})^x$.
112. $5^{t \ln t}$.
113. $7^{\frac{1}{x}}$.
114. $a^x \ln x$.
115. $\ln \sin^n x$.
116. $\ln \ln x$.
117. $\ln \left(\frac{1 - e^x}{e^x}\right)$.
118. $\frac{1}{a} \tan^{-1} \frac{1}{a} + \frac{1}{2} \ln (a^2 + x^2)$.
119. $e^{-kt} \cos (a + bt)$.
120. $\ln (a + \sqrt{a^2 + x^2})$.
121. $\left(x + \frac{1}{x}\right) \ln \left(x + \frac{1}{x}\right) - x - \frac{1}{x}$.
122. $\ln \frac{\sqrt{x + a} + \sqrt{x - a}}{\sqrt{x + a} - \sqrt{x - a}}$.
123. $\tan^{-1} \frac{1}{2}(e^x + e^{-x})$.
124. $\ln (\sqrt{x} + \sqrt{x + 2})$.

125. $(x + 1) \ln (x^2 + 2x + 5) + \frac{3}{2} \tan^{-1} \frac{x + 1}{2}$.
126. $\sec \frac{1}{2} x \tan \frac{1}{2} x + \ln (\sec \frac{1}{2} x + \tan \frac{1}{2} x)$.

127. $x \sec^{-1} \frac{1}{2}\left(x + \frac{1}{x}\right) - \ln(x^2 + 1)$.

128. $\frac{x}{3} \ln\left(\frac{4}{9}x^2 + 1\right) - \frac{2}{3}x + \tan^{-1}\frac{2}{3}x$.

CHAPTER VII

Find the equations of the tangent and normal to each of the following curves at the point indicated:

129. $y^2 = 2x + y$, at $(1, 2)$.
130. $x^2 - y^2 = 5$, at $(3, 2)$.
131. $x^2 + y^2 = x + 3y$, at $(-1, 1)$.
132. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 2$, at $(1, 1)$.
133. $y = \ln x$, at $(1, 0)$.
134. $x^2(x + y) = a^2(x - y)$, at $(0, 0)$.
135. $x = 2\cos\theta,\ y = 3\sin\theta$, at $\theta = \frac{\pi}{2}$.
136. $r = a(1 + \cos\theta)$, at $\theta = \frac{\pi}{4}$.

Find the angles at which the following pairs of curves intersect:

137. $x^2 + y^2 = 8x,\ y^2(2 - x) = x^3$.
138. $y^2 = 2ax + a^2,\ x^2 = 2by + b^2$.
139. $x^2 = 4ay,\ (x^2 + 4a^2)y = 8a^3$.
140. $y^2 = 6x,\ x^2 + y^2 = 16$.
141. $y = \frac{1}{2}(e^x + e^{-x}),\ y = 1$.
142. $y = \sin x,\ y = \sin 2x$.
143. Show that all the curves obtained by giving different values to n in the equation

$$\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2,$$

are tangent at (a, b).

144. Show that for all values of a and b the curves

$$x^3 - 3xy^2 = a, \quad 3x^2y - y^3 = b,$$

intersect at right angles.

Examine each of the following curves for direction of curvature and points of inflection:

145. $y = \dfrac{1 - x}{1 + x^2}$.
146. $y = \tan x$.
147. $x = 6y^2 - 2y^3$.

148. $x = 2t - \frac{1}{t^2}$, $y = 2t + \frac{1}{t^2}$.

149. Clausius's equation connecting the pressure, volume, and temperature of a gas is

$$p = \frac{RT}{v - a} - \frac{c}{T(v + b)^3},$$

R, a, b, c being constants. If T is constant and p, v the coördinates of a point, this equation represents an isothermal. Find the value of T for which the tangent at the point of inflection is horizontal.

150. If two curves $y = f(x)$, $y = F(x)$ intersect at $x = a$, and $f'(a) = F'(a)$, but $f''(a)$ is not equal to $F''(a)$, show that the curves are tangent and do not cross at $x = a$. Apply to the curves $y = x^2$ and $y = x^3$ at $x = 0$.

151. If two curves $y = f(x)$, $y = F(x)$ intersect at $x = a$, aud $f'(a) = F'(a)$, $f''(a) = F''(a)$, but $f'''(a)$ is not equal to $F'''(a)$, show that the curves are tangent and cross at $x = a$. Apply to the curves $y = x^2$ and $y = x^2 + (x - 1)^3$ at $x = 1$.

Find the radius of curvature on each of the following curves at the point indicated:

152. Parabola $y^2 = ax$ at its vertex.

153. Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at its vertices.

154. Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $x = \sqrt{a^2 + b^2}$.

155. $y = \ln \csc x$, at $\left(\frac{\pi}{2}, 0\right)$.

156. $x = \frac{1}{2}\sin y - \frac{1}{2}\ln(\sec y + \tan y)$, at any point (x, y).

157. $x = a\cos^3\theta$, $y = a\sin^3\theta$, at any point.

158. Find the center of curvature of $y = \ln(x - 2)$ at $(3, 0)$.

Find the angle ψ at the point indicated on each of the following curves:

159. $r = 2^\theta$, at $\theta = 0$.

160. $r = a + b\cos\theta$, at $\theta = \frac{\pi}{2}$.

161. $r(1 - \cos\theta) = k$, at $\theta = \frac{\pi}{3}$.

162. $r = a\sin 2\theta$, at $\theta = \frac{\pi}{4}$.

Find the angles at which the following pairs of curves intersect:

163. $r(1 - \cos\theta) = a$, $\quad r = a(1 - \cos\theta)$.

164. $r = a\sec^2\frac{\theta}{2}$, $\quad r = b\csc^2\frac{\theta}{2}$.

165. $r = a \cos \theta$, $\qquad r = a \cos 2\theta$.
166. $r = a \sec \theta$, $\qquad r = 2a \sin \theta$.

Find the equations of the tangent lines to the following curves at the points indicated:

167. $x = 2t, y = \frac{2}{t}, z = t^2$, at $t = 2$.
168. $x = \sin t$, $y = \cos t$, $z = \sec t$, at $t = 0$.
169. $x^2 + y^2 + z^2 = 6$, $x + y + z = 2$, at $(1, 2, -1)$.
170. $z = x^2 + y^2$, $z^2 = 2x - 2y$, at $(1, -1, 2)$.

CHAPTER VIII

171. A point describes a circle with constant speed. Show that its projection on a fixed diameter moves with a speed proportional to the distance of the point from that diameter.

172. The motion of a point (x, y) is given by the equations

$$x = \frac{t}{2}\sqrt{a^2 - t^2} + \frac{a^2}{2}\sin^{-1}\frac{t}{a},$$

$$y = \frac{t}{2}\sqrt{a^2 + t^2} + \frac{a^2}{2}\ln\left(t + \sqrt{a^2 + t^2}\right).$$

Show that its speed is constant.

Find the speed, velocity, and acceleration in each of the following cases:

173. $x = 2 + 3t$, $y = 4 - 9t$.
174. $x = a\cos(\omega t + \alpha)$, $y = a \sin(\omega t + \alpha)$.
175. $x = a + \alpha t$, $y = b + \beta t$, $z = c + \gamma t$.
176. $x = e^t \sin t$, $y = e^t \cos t$, $z = kt$.

177. The motion of a point $P\ (x, y)$ is determined by the equations

$$x = a\cos(nt + \alpha),\ y = b\sin(nt + \alpha).$$

Show that its acceleration is directed toward the origin and has a magnitude proportional to the distance from the origin.

178. A particle moves with constant acceleration along the parabola $y^2 = 2cx$. Show that the acceleration is parallel to the x-axis.

179. A particle moves with acceleration $[a, o]$ along the parabola $y^2 = 2cx$. Find its velocity.

180. Show that the vector $\left[\frac{d^2x}{ds^2}, \frac{d^2y}{ds^2}\right]$ extends along the normal at (x, y) and is in magnitude equal to the curvature at (x, y).

CHAPTER IX

181. Show that the function

$$x^{\frac{2}{3}} - 1$$

vanishes at $x = -1$ and $x = 1$, but that its derivative does not vanish between these values. Is this an exception to Rolle's theorem?

182. Show that the equation

$$x^5 - 5x + 4 = 0$$

has only two distinct real roots.

183. Show that

$$\lim_{x \doteq 0} \frac{x^2 \sin \frac{1}{x}}{\sin x} = 0,$$

but that this value cannot be found by the methods of Art. 73. Explain.

184. Show that

$$\lim_{x \doteq 0} \frac{1 - \cos x}{\cos x} = 0.$$

Why cannot this result be obtained by the methods of Art. 73?

Find the values of the following limits:

185. $\lim\limits_{x \doteq 0} \dfrac{xe^{3x} - x}{1 - \cos 2x}$.

186. $\lim\limits_{x \doteq 3} \dfrac{\sqrt{3x} - \sqrt{12 - x}}{2x - 3\sqrt{19 - 5x}}$.

187. $\lim\limits_{x \doteq 1} \dfrac{\tan \frac{\pi x}{2}}{1 + \csc(x - 1)}$.

188. $\lim\limits_{x \doteq 1} \dfrac{\ln(1 - x)}{\cot(\pi x)}$.

189. $\lim\limits_{x \doteq 0} \left(\dfrac{1}{x^2} - \dfrac{\cot x}{x}\right)$.

190. $\lim\limits_{x \doteq \infty} x^n e^{-x^2}$.

191. $\lim\limits_{x \doteq 0} \dfrac{x \ln x}{\sin^2 x - x \cot x}$.

192. $\lim\limits_{x \doteq 0} (\sec x)^{\frac{1}{x^2}}$.

193. The area of a regular polygon of n sides inscribed in a circle of radius a is

$$na^2 \sin \frac{\pi}{n} \cos \frac{\pi}{n}.$$

Show that this approaches the area of the circle when n increases indefinitely.

194. Show that the curve

$$x^3 + y^3 = 3xy$$

is tangent to both coördinate axes at the origin.

CHAPTER X

Determine the values of the following functions correct to four decimals:

195. $\cos 62°$.

196. $\sin 33°$.

197. $\ln (1.2)$.

198. $\sqrt[5]{1.1}$.

199. $\tan^{-1} (\frac{1}{10})$.

200. $\csc (31°)$.

201. Calculate π by expanding $\tan^{-1} x$ and using the formula

$$\frac{\pi}{4} = \tan^{-1} (1).$$

202. Given $\ln 5 = 1.6094$, calculate $\ln 24$.

203. Prove that

$$D = \sqrt{\tfrac{3}{2} h}$$

is an approximate formula for the distance of the horizon, D being the distance in miles and h the altitude of the observer in feet.

Prove the following expansions indicating if possible the values of x for which they converge:

204. $\ln (1 + x^2) = \ln 10 + \frac{3}{5} (x - 3) - \frac{2}{25} (x - 3)^2 + \cdots$.

205. $\ln (e^x + e^{-x}) = \dfrac{x^2}{2} - \dfrac{x^4}{12} + \dfrac{x^6}{45} + \cdots$.

206. $\ln (1 + \sin x) = x - \frac{1}{2} x^2 + \frac{1}{6} x^3 - \frac{1}{12} x^4 + \cdots$.

207. $e^x \sec x = 1 + x + x^2 + \frac{2}{3} x^3 + \frac{1}{2} x^4 + \cdots$.

208. $\ln \left(x + \sqrt{1 + x^2}\right) = x - \dfrac{1}{2}\dfrac{x^3}{3} + \dfrac{1 \cdot 3}{2 \cdot 4}\dfrac{x^5}{5} + \cdots$.

209. $\ln \dfrac{x + 1}{x - 1} = 2 \left[\dfrac{1}{x} + \dfrac{1}{3 x^3} + \dfrac{1}{5 x^5} + \cdots\right]$.

210. $\ln \tan x = \ln x + \dfrac{x^2}{3} + \dfrac{7 x^4}{90} + \cdots$.

211. $e^{\sin x} = 1 + x + \dfrac{x^2}{2!} - \dfrac{3 x^4}{4!} - \cdots$.

212. $e^{\tan x} = 1 + x + \dfrac{x^2}{2!} + \dfrac{3 x^3}{3!} + \dfrac{9 x^4}{4!} + \cdots$.

Determine the values of x for which the following series converge:

213. $1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^4}{4} + \cdots$.

214. $(x - 1) + \dfrac{(x - 1)^2}{2^2} + \dfrac{(x - 1)^3}{3^3} + \dfrac{(x - 1)^4}{4^4} + \cdots$.

215. $1 + 2 x + 3 x^2 + 4 x^3 + \cdots$.

216. $2 + \dfrac{x + 2}{1 \cdot 2} + \dfrac{(x + 2)^2}{2 \cdot 3} + \dfrac{(x + 2)^3}{3 \cdot 4} + \cdots$.

CHAPTER XI

In each of the following exercises show that the partial derivatives satisfy the equation given:

217. $u = xy + y^2z^2, \quad x\frac{\partial u}{\partial x} + z\frac{\partial u}{\partial z} = y\frac{\partial u}{\partial y}.$

218. $z = x^4 - 2x^2y^2 + y^4, \quad y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0.$

219. $u = (x + y)\ln xz, \quad x\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) = z\frac{\partial u}{\partial z}.$

220. $u = \left(x + \frac{1}{y}\right)\tan^{-1}\left(y - \frac{1}{z}\right), \quad \frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} = y^2z^2\frac{\partial u}{\partial z}.$

221. $u = xy + \frac{z}{x},$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xz\frac{\partial^2 u}{\partial x \partial z} + z^2\frac{\partial^2 u}{\partial z^2} = y^2\frac{\partial^2 u}{\partial y^2}.$$

222. $z = \ln(x^2 + y^2), \quad \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$

223. $u = \frac{y + x}{y - z},$

$$\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial x^2}.$$

Prove the following relations assuming that z is a function of x and y:

224. $u = (x + y - z)^2,$

$$\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}\frac{\partial z}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial z}{\partial y}.$$

225. $u = z + e^{xy},$

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

226. $u = z(x^2 - y^2),$

$$y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = (x^2 - y^2)\left(y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}\right).$$

227. If $x = \frac{r}{2}(e^\theta + e^{-\theta}), \quad y = \frac{r}{2}(e^\theta - e^{-\theta})$, show that

$$\left(\frac{\partial x}{\partial r}\right)_\theta = \left(\frac{\partial r}{\partial x}\right)_y.$$

228. If $xyz = a^3$, show that

$$\left(\frac{\partial y}{\partial x}\right)_z\left(\frac{\partial z}{\partial y}\right)_x\left(\frac{\partial x}{\partial z}\right)_y = -1.$$

In each of the following exercises find Δz and its principal part, assuming that x and y are the independent variables. When Δx and Δy approach zero, show that the difference of Δz and its principal part is an infinitesimal of higher order than Δx and Δy.

229. $z = xy$.

230. $z = x^2 - y^2 + 2x$.

231. $z = \dfrac{y}{x^2 + 1}$.

232. $z = \sqrt{x^2 + y^2}$.

Find the total differentials of the following functions:

233. $ax^4 + bx^2y^2 + cy^4$.

234. $\ln(x^2 + y^2 + z^2)$.

235. $x^2 \tan^{-1}\dfrac{y}{x} - y^2 \tan^{-1}\dfrac{x}{y}$.

236. $yze^x + zxe^y + xye^z$.

237. If $u = x^n f(z)$, $z = \dfrac{y}{x}$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

238. If $u = f(r, s)$, $r = x + y$, $s = x - y$, show that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial r}.$$

239. If $u = f(r, s, t)$, $r = \dfrac{x}{y}$, $s = \dfrac{y}{z}$, $t = \dfrac{z}{x}$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$$

240. If α is the angle between the x-axis and the line OP from the origin to $P(x, y, z)$, find the derivatives of α in the directions parallel to the coördinate axes.

241. Show that

$$(\cot y - y \sec x \tan x)\,dx - (x \csc^2 y + \sec x)\,dy$$

is an exact differential.

Find the equations of the normal and tangent plane to each of the following surfaces at the point indicated:

242. $x^2 + 2y^2 - z^2 = 16$, at $(3, 2, -1)$.

243. $2x + 3y - 4z = 4$, at $(1, 2, 1)$.

244. $z^2 = 8xy$, at $(2, 1, -4)$.

245. $y = z^2 - x^2 + 1$, at $(3, 1, -3)$.

246. Show that the largest rectangular parallelopiped with a given surface is a cube.

247. An open rectangular box is to be constructed of a given amount of material. Find the dimensions if the capacity is a maximum.

248. A body has the shape of a hollow cylinder with conical ends. Find the dimensions of the largest body that can be constructed from a given amount of material.

249. Find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

250. Show that the triangle of greatest area inscribed in a given circle is equilateral.

251. Find the point so situated that the sum of its distances from the vertices of an acute angled triangle is a minimum.

252. At the point (x, y, z) of space find the direction along which a given function $F(x, y, z)$ has the largest directional derivative.

ANSWERS TO EXERCISES

Page 8

1. $-\frac{8}{5}$.
2. $\sqrt{2}$.
3. -1.
4. -1.
5. 1.
6. $\frac{1}{2}$.

Page 14

3. 2.
5. The tangents are parallel to the x-axis at $(-1, -1)$, $(0, 0)$, and $(1, -1)$. The slope is positive between $(-1, -1)$ and $(0, 0)$ and on the right of $(1, -1)$.
10. Negative.
11. Positive in 1st and 4th quadrants, negative in 2nd and 3rd.

Pages 27, 28

31. When $x = 4$, $y = \frac{4}{5}$ and $dy = 0.072\,dx$. When x changes to 4.2, $dx = 0.2$ and an approximate value for y is $y + dy = 0.814$. This agrees to 3 decimals with the exact value.
32. When $x = 0$, the function is equal to 1 and its differential is $-dx$. When $x = 0.3$, an approximate value is then $1 - dx = 0.7$. The exact value is 0.754.
34. 18.
35. $(a, 2\,a)$.
36. Increases when $x < \dfrac{a}{3}$, decreases when $x > \dfrac{a}{3}$.
37. $x = \pm \dfrac{1}{\sqrt[4]{2}}$.
38. $\tan^{-1} \frac{2}{3}$.
39. $-\dfrac{2}{(x-1)^2} = -\dfrac{(y-2)^2}{2}$.

Page 31

1. $-\dfrac{2}{(x-1)^2}$, $\dfrac{4}{(x-1)^3}$.
2. $-\dfrac{x}{\sqrt{a^2 - x^2}}$, $-\dfrac{a^2}{(a^2 - x^2)^{\frac{3}{2}}}$.
3. $(x-1)^2\,(x+2)^3\,(7\,x+2)$, $(x-1)\,(x+2)^2\,(42\,x^2 + 24\,x - 12)$.
4. $\dfrac{2}{y}$, $-\dfrac{4}{y^3}$.
5. $-\dfrac{x}{y}$, $-\dfrac{a^2}{y^3}$.
6. $\dfrac{x}{2\,y}$, $-\dfrac{1}{4\,y^3}$.
7. $\dfrac{1-y}{x-1}$, $\dfrac{2}{(x-1)^3}$.
8. $-\dfrac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$, $\dfrac{a^{\frac{2}{3}}}{3\,x^{\frac{4}{3}}\,y^{\frac{1}{3}}}$.

13. $\dfrac{d^2y}{dx^2} = -\dfrac{12}{t\,(2+3\,t)^3}, \quad \dfrac{d^2x}{dy^2} = \dfrac{12}{t\,(2-3\,t)^3}.$

Pages 36–38

1. $v = 100 - 32\,t,\ a = -32$. Rises until $t = 3\frac{1}{8}$. Highest point $h = 206.25$.
2. $v = t^3 - 12\,t^2 + 32\,t,\ a = 3\,t^2 - 24\,t + 32$. Velocity decreasing between $t = 1.691$ and $t = 6.309$. Moving backward when t is negative or between 4 and 8.
7. $\omega = b - 2\,ct,\ \alpha = -2\,c$. Wheel comes to rest when $t = \dfrac{b}{2\,c}$.
10. $9\,\pi$ cu. ft./min.
11. $144\,\pi$ sq. ft./sec.
12. Decreasing $8\,\pi$ cu. ft./sec.
13. $\sqrt{3} : 1$.
14. $\frac{1}{3}\sqrt{3}$ in./sec.
15. $12\frac{1}{2}$ ft./sec., $7\frac{1}{2}$ ft./sec.
16. $4\sqrt{3}$ mi./hr.
17. $\dfrac{c \tan \beta}{\pi a^2}$ ft./sec.
18. Neither approaching nor separating.
19. 25.8 ft./sec.
20. $64\sqrt{3}$ ft./sec.

Pages 43–45

1. Minimum $3\frac{7}{8}$.
2. Minimum -10, maximum 22.
3. Maximum at $x = 0$, minima at $x = -1$ and $x = +1$.
4. Minimum when $x = 0$.
10. $\frac{3}{2}\sqrt[3]{2}$.
13. $\frac{2}{3}\,a\sqrt{3}$.
14. Length of base equals twice the depth of the box.
16. Radius of base equals two-thirds of the altitude.
17. Altitude equals $\dfrac{4}{\pi}$ times diameter of base.
18. $\dfrac{16\,\pi\sqrt{3}}{27}$.
19. $\frac{1}{4}\,(a_1 + a_2 + a_3 + a_4)$.
20. Girth equals twice length.
21. Radius equals $2\sqrt{6}$ inches.
22. The distance from the more intense source is $\sqrt[3]{2}$ times the distance from the other source.
23. $12\sqrt{2}$.
24. $[5^{\frac{2}{3}} + 6^{\frac{2}{3}}]^{\frac{3}{2}}$.
25. $19\frac{1}{3}$ ft.
26. Radius of semicircle equals height of rectangle.
27. 4 pieces 6 inches long and 2 pieces 2 ft. long.
28. The angle of the sector is two radians.
29. At the end of 4 hours.

30. He should land 4.71 miles from his destination.

31. $\dfrac{a\sqrt{2}}{2}$, a being the length of side.

35. $2\frac{1}{2}$ mi./hr.

36. 13.6 knots.

Page 48

1. Maximum $= a$, minimum $= -a$.
2. Maximum $= 0$, minimum $= -\left(\frac{4}{27}\right)^{\frac{1}{3}}$.
3. Minimum $= -1$.
4. Minimum $= 0$, maximum $= \frac{4}{27}$.

10. Either 4 or 5.

Pages 52, 53

19. $A = 3$.

20. $A = -\frac{7}{68}$, $B = -\frac{3}{34}$.

21. $\sqrt{3} - \frac{3}{2}$.

22. $n\pi \pm \dfrac{\pi}{6}$, n being any integer.

23. Velocity $= -2\pi nA$, acceleration $= 0$.

24. $\dfrac{40\pi}{3}$ miles per minute.

25. $\frac{9}{4}$ radians per hour.

26. $\frac{9}{2} + \frac{5}{6}\pi\sqrt{3}$.

28. $13\sqrt{13}$.

29. The needle will be inclined to the horizontal at an angle of about $32° 30'$

30. 120°.

31. 120°.

32. $\dfrac{a}{\pi}$.

33. If the spokes are extended outward, they will form the sides of an isosceles triangle.

Page 56

24. $\omega = \dfrac{v}{r}\cos\phi$, r being the radius of pulley and ϕ the angle formed by the string and line along which its end moves.

25. $4\sqrt{35}$.

Page 61

27. $x = n\pi + \cot^{-1} 2$, n being any integer.

30. $x < -3$, $x > 2$, or $-2 < x < 1$.

Pages 65, 66

1. $2y - x = 5$, $y + 2x = 0$.
2. $y + 4x = 8$, $4y - x = 15$.
3. $2y \mp x = \pm a$, $y \pm 2x = \pm 3a$.

4. $y = a(x \ln b + 1), \quad x + ay \ln b = a^2 \ln b.$

5. $y - \frac{3}{2} = \frac{3}{2}\sqrt{3}\left(x - \frac{\pi}{6}\right), \quad y - \frac{3}{2} + \frac{2}{9}\sqrt{3}\left(x - \frac{\pi}{6}\right) = 0.$

6. $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1, \quad b^2x_1y - a^2y_1x = (b^2 - a^2)\,x_1y_1.$

7. $x + y = 2, \quad x - y = 2.$

8. $x + 3y = 4, \quad y - 3x = 28.$

9. $y + x \tan \frac{1}{2}\phi_1 = a\phi_1 \tan \frac{1}{2}\phi_1.$

10. $90°, \quad \tan^{-1} \frac{3}{4}.$

11. $45°.$

12. $90°.$

13. $\tan^{-1} 2\sqrt{2}.$

14. $\tan^{-1} \frac{\ln 10 - 1}{\ln 10 + 1}.$

15. $\tan^{-1} 3\sqrt{3}.$

Page 70

1. Point of inflection (0, 3). Concave upward on the right of this point, downward on the left.
2. Point of inflection $(\frac{1}{2}, -\frac{5}{2})$. Concave upward on the right of this point, downward on the left.
3. The curve is everywhere concave upward. There is no point of inflection.
4. Point of inflection (1, 0). Concave on the left of this point, downward on the right.
5. Point of inflection $\left(-2, -\frac{2}{e^2}\right)$. Concave upward on the right, downward on the left.
6. Points of inflection at $x = \pm\frac{1}{\sqrt{2}}$. Concave downward between the points of inflection, upward outside.
7. Points of inflection (0, 0), (± 3, ± 1). Concave upward when $-3 < x < 0$ or $x > 3$.
8. Point of inflection at the origin. Concave upward on the left of the origin.

Pages 76, 77

1. $\pm 2\sqrt{6}.$

3. $\frac{a^2}{b}.$

4. $3\sqrt{2}.$

5. $e^{\frac{\pi}{2}}\sqrt{2}.$

6. $\frac{4}{3}a.$

7. $\frac{y^2}{a}.$

8. $\sec y.$

9. $\frac{(y^2+1)^2}{4y}.$

10. $2a\sec^3\frac{\theta}{2}.$

Page 79

There are two angles ψ depending on the direction in which s is measured along the curve. In the following answers only one of these angles is given.

1. $\tan^{-1}\left(\frac{\pi}{3}\right)$.
2. $\frac{\pi}{4}$.
3. $\frac{\pi}{3}$.
4. $\tan^{-1}(-2)$.
5. $\frac{1}{6}\pi$.
7. $0°$, $90°$, and $\tan^{-1} 3\sqrt{3}$.
8. $\theta = \pm\frac{2}{3}\pi$.
10. 3.

Page 84

1. $\dfrac{x-\sqrt{2}}{\sqrt{2}} = \dfrac{y-1}{2} = \dfrac{z-\frac{\pi a}{4}}{a}$.
2. $\dfrac{x-e}{e} = 1 - ye = \dfrac{z-1}{2}$.
3. $\dfrac{x-e^{\frac{\pi}{2}}}{e^{\frac{\pi}{2}}} = -\dfrac{y}{e^{\frac{\pi}{2}}} = \dfrac{z-\frac{\pi k}{2}}{k}$.
5. $\tan^{-1}\frac{a}{k}$.
6. $\tan^{-1}\frac{t}{\sqrt{2}}$.
7. $69° 29'$.

Pages 92, 93

1. The angular speed is $\dfrac{av}{a^2+x^2}$, where x is the abscissa of the moving point.
2. If x_1 is the abscissa of the end in the x-axis and y_1 the ordinate of the end in the y-axis, the velocity of the middle point is

$$\left[\pm\tfrac{1}{2}v, \quad \mp\frac{vx_1}{2y_1}\right],$$

the upper signs being used if the end in the x-axis moves to the right, the lower signs if it moves to the left. The speed is $\dfrac{av}{2y_1}$.

3. The velocity is

$$[v - a\omega\sin\theta, \quad a\omega\cos\theta],$$

where θ is the angle from the x-axis to the radius through the moving point. The speed is

$$\sqrt{v^2 + a^2\omega^2 - 2a\omega v\sin\theta}.$$

6. The boat should be pointed 30° up the river.
7. Velocity $= [a, b, c - gt]$, Acceleration $= [0, 0, -g]$,

$$\text{Speed} = \sqrt{a^2 + b^2 + (e - gt)^2}.$$

9. Velocity $= [a\omega(1 - \cos\phi), \quad a\omega \sin\phi]$,
Speed $= a\omega\sqrt{2 - 2\cos\phi} = 2\,a\omega \sin \tfrac{1}{2}\phi$,
Acceleration $= [a\omega^2 \sin\phi, \quad a\omega^2 \cos\phi]$.

10. $\left[-\dfrac{3\,v^2}{4\,a}\dfrac{\sin\frac{3}{2}\theta}{\sin\frac{1}{2}\theta}, \quad \dfrac{3\,v^2}{4\,a}\dfrac{\cos\frac{3}{2}\theta}{\sin\frac{1}{2}\theta}\right]$.

12. $x = vt\cos\omega t, \quad y = vt\sin\omega t$. The velocity is the sum of the partial velocities, but the acceleration is not.

13. $x = a\cos\omega t + b\cos 2\,\omega t$, $y = a\sin\omega t + b\sin 2\,\omega t$. The velocity is the sum of the partial velocities and the acceleration the sum of the partial accelerations.

14. $x = a\omega_1 t - a\sin(\omega_1 + \omega_2)\,t, \quad y = a\cos(\omega_1 + \omega_2)\,t$. The velocity is the sum of the partial velocities and the acceleration the sum of the partial accelerations.

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2. $\frac{9}{10}$.
3. n.
4. 0.
5. e^a.
6. 2.
7. -2.
8. 1.
9. 0.
10. π^2.
11. 1.
12. 1.
13. 0.
14. $-\frac{1}{2}$.
15. 0.
16. 0.
17. $-\frac{1}{3}$.
18. $\frac{1}{2}$.
19. 1.
20. -3.
21. a.
22. $\dfrac{1}{\pi}$.
23. $f'(x)\,dx$.
24. ∞.
25. $\frac{1}{2}$.
26. ∞.
27. ∞.
28. 1.
29. 1.
30. a.
31. e^m

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1. 0.0872.
2. 0.8480.
3. 1.0724.
4. 1.6003.
5. 1.0154.
6. 0.1054.
7. 1.6487.
8. 0.0997.
9. 2.833.

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21. $(-2, 1, 0)$.
22. $(1, 1, 2)$.

Pages 123, 124

1. Increment $= -0.151$, principal part $= -0.154$.
4. $\dfrac{dT}{T} = \dfrac{1}{2}\left(\dfrac{dl}{l} - \dfrac{dg}{g}\right)$. Since dl and dg may be either positive or negative, the percentage error in T may be $\frac{1}{2}$ the sum of the percentage errors in l and g.
5. The percentage error in g may be as great as that in s plus twice that in T.

13. $-\dfrac{u+v}{u}$.

15. $\dfrac{2z^2 - uy}{zx - 2uv}$.

14. $\dfrac{1}{x}(x^2 + y^2 + xy - z^2)$.

Pages 131, 132

1. $\dfrac{du}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{d\phi}{dx}$.

2. $\left(\dfrac{\partial u}{\partial x}\right)_y = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial z}\dfrac{d\phi}{dx}$.

3. $\dfrac{du}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{d\psi}{\partial x} + \dfrac{\partial f}{\partial z}\left(\dfrac{\partial \phi}{\partial x} + \dfrac{\partial \phi}{\partial y}\dfrac{d\psi}{dx}\right)$.

5. $\dfrac{dz}{dx} = \dfrac{\dfrac{\partial f}{\partial y}\dfrac{\partial F}{\partial x} - \dfrac{\partial f}{\partial x}\dfrac{\partial F}{\partial y}}{\dfrac{\partial f}{\partial y} + \dfrac{\partial f}{\partial z}\dfrac{\partial F}{\partial y}}$.

13. $3\sqrt{3} - 4$.

14. $-\dfrac{e}{r^3}(x\cos\alpha + y\cos\beta + z\cos\gamma)$.

Page 135

1. $\dfrac{x-1}{1} = \dfrac{y-2}{2} = \dfrac{z-2}{2}$, $(x-1) + 2(y-2) + 2(z-2) = 0$.
2. Normal, $y + 4x = 5$, $z = 3$.
   Tangent plane, $x - 4y - 14 = 0$.
3. $\dfrac{x-3}{3} = \dfrac{y-4}{4} = \dfrac{z-5}{-5}$. $3x + 4y - 5z = 0$.
4. $\dfrac{x-5}{1} = \dfrac{y-1}{5} = \dfrac{z-3}{-3}$, $x + 5y - 3z - 1 = 0$.
5. $\dfrac{x-5}{-1} = \dfrac{y-1}{4} = \dfrac{z-1}{6}$. $x - 4y - 6z + 5 = 0$.
6. $x + z = y - z = \pm\sqrt{2}$.

Pages 138, 139

1. The box should have a square base with side equal to twice the depth.
2. The cylinder and cone have volumes in the ratio 3 : 2 and lateral surfaces in the ratio 2 : 3.
4. The center of gravity of the triangle ABC.

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